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Abstract

A functional analytic framework is proposed for setting up the variational formulation of the stationary, thermal-electro hydrodynamical Boussinesq equations. In this setting, existence, stability and uniqueness of solutions in a suitable Sobolev space is shown. The results are obtained by extending the existing theory on stationary Boussinesq equations to take into account a more general force term and by employing a fixed-point argument for augmenting the Boussinesq equations with Gauss' law and the dielectrophoretic force.

1 Introduction

Gravitation acting on non-isothermal fluids induces the well-known buoyancy force which gives rise to a variety of different flow structures. An important application of this effect is a heat exchanging system, where a fluid acts as transmitter of thermal energy between a hot object and a cooling device. The convection of temperature due to vortex-like fluid structures plays the major role in heat transfer. To further enhance heat transfer or to make it possible in non-gravitational environments, e.g. in space devices in earth orbit, without employing mechanical devices such as pumps or rotors, one can make use of the so called *dielectrophoretic* (DEP) force. The DEP force acts on dielectric fluids under the influence of an outer electrical field with direction determined by the fluid's temperature gradient. The resulting system of partial differential equations is obtained by augmenting the standard Boussinesq equations for natural convection for small temperature variations with the DEP force in the momentum equation and with Gauss's law for describing the electric potential inside the fluid. This system is called *Thermal-Electro Hydrodynamic (TEHD) Boussinesq Equations* and is given by [20]

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \frac{1}{\rho_r}\nabla p &= \alpha_e(\nabla\Phi)^2\nabla\theta - \alpha_g \mathbf{g}(\theta-\theta_r) \\ \nabla\cdot\mathbf{u} &= 0 \\ \partial_t\theta + (\mathbf{u}\cdot\nabla)\theta - \kappa\Delta\theta &= 0 \\ -\nabla\cdot(\epsilon_r(1-\gamma(\theta-\theta_r))\nabla\Phi) &= 0. \end{aligned} \tag{1.1}$$

The DEP force $\mathbf{f}_e := \alpha_e(\nabla\Phi)^2\nabla\theta$ is derived from the more general electrical body force

$$\mathbf{F}_E = \rho_e\mathbf{E} - \frac{1}{2}\mathbf{E}^2\nabla\epsilon(\theta) + \frac{1}{2}\nabla\left[\rho\left(\frac{\partial\epsilon}{\partial\rho}\right)_\theta \mathbf{E}^2\right]. \tag{1.2}$$

Here, the Coulomb force $\rho_e\mathbf{E}$ can be neglected if the electric field $\mathbf{E}$ is induced by an alternating voltage of high frequency [20] and the third term can be hidden inside a generalized pressure since it is a gradient field. Then, $\mathbf{f}_e$ is obtained from the second term in (1.2) by linearizing the temperature dependent permittivity $\epsilon(\theta)$ around some reference temperature θ_r, i.e.

$$\epsilon(\theta) = \epsilon_r(1-\gamma(\theta-\theta_r)) \text{ and } \alpha_e = \frac{\epsilon_r\gamma}{2\rho_r}.$$

The set of unknown variables consists of fluid velocity $\mathbf{u}$, pressure p, temperature θ and potential Φ. The physical parameters are kinematic viscosity ν, density ρ_r, thermal diffusion κ, thermal expansion coefficient α_g, natural gravity $\mathbf{g}$, reference permittivity ϵ_r and permittivity rate of change γ. The system (1.1) is posed on a bounded domain $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ and subjected to the following boundary conditions

$$\begin{aligned} \mathbf{u} &= 0 \text{ on } \Gamma \\ \theta &= \theta_D \text{ on } \Gamma_D,\ \nabla\theta\cdot\mathbf{n} = 0 \text{ on } \Gamma_N \\ \Phi &= \Phi_D \text{ on } \Gamma_D,\ \nabla\Phi\cdot\mathbf{n} = 0 \text{ on } \Gamma_N \end{aligned}$$

with $\Gamma := \partial\Omega = \Gamma_D + \Gamma_N$.

In this work, we restrict ourselves to the stationary version of the TEHD equations, which might be of interest on its own or which are obtained after discretizing (1.1) in time. The considered system is given by

$$\begin{aligned} \delta\mathbf{u} + (\bar{\mathbf{u}}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \frac{1}{\rho_r}\nabla p &= \mathbf{F}(\theta,\bar{\Phi}) + \mathbf{f_v} \\ \nabla\cdot\mathbf{u} &= 0 \\ \delta\theta + (\tilde{\mathbf{u}}\cdot\nabla)\theta - \kappa\Delta\theta &= f_\tau \\ -\nabla\cdot(\epsilon(\bar{\theta})\nabla\Phi) &= f_\beta(\theta), \end{aligned} \tag{1.3}$$

with $\delta \geq 0$ and $\mathbf{f_v}, f_\tau, f_\beta$ denoting possible contributions by some outer time-stepping scheme, which also determines $(\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi})$. Depending on the degree of implicitness, each of these variables could be either fixed or unknown. In particular, we allow the case $(\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}) = (\mathbf{u}, \mathbf{u}, \theta, \Phi)$, $\delta = 0$ and $\mathbf{f_v} = f_\tau = f_\beta = 0$, which corresponds to the stationary version of the transient TEHD equations (1.1). Compared to (1.1), we introduced a general force term $\mathbf{F}(\theta, \bar{\Phi})$, a general permittivity $\epsilon\colon \mathbb{R} \to \mathbb{R}$ and shifted θ by the constant reference temperature θ_r.

The effect of the DEP force has been first studied theoretically and experimentally in [4] and theoretically in [23]. In the recent years, there have been a number of works concerning linear stability analysis (LSA) of the TEHD equations in various settings. In [28], [27] and [15] LSA was performed in case of infinite length plate and cylinder annulus geometry in absence of natural gravity, i.e. with $\mathbf{g} = 0$. Corresponding experiments under microgravity conditions were conducted in [6] and [17]. In [17], the experimental data was compared with LSA results. Experiments under the influence of earth gravity were performed in [9] and [22]. In [22], experimental data was compared with direct numerical simulation based on the Finite Element method. The numerical solution of the TEHD equations was also considered by employing a Finite Volume method in [29], [13] and by using Finite Elements in [5] and [8]. In [29], a spherical gap was considered and the effect of dielectric heating was taken into account. The other works focused on vertical cylinder annuli. In [25] and [26], periodic top and bottom plates of the cylinder were assumed, making it possible to employ spectral methods for numerical simulation. A more comprehensive overview on DEP-driven flow is given by the review paper [20].

To our best knowledge, there are no contributions that address the functional analytic investigation of both the stationary and instationary TEHD Boussinesq equations. In this work, we therefore investigate stability, existence and uniqueness of solutions of the stationary TEHD equations. In doing so, we extend the theoretical work on the stationary Boussinesq equations performed in [19] by including a more general body force term.

We aim to put the stationary TEHD equations (1.3) into a functional analytic framework that allows to investigate existence, uniqueness and stability of solutions. In doing so, certain difficulties are encountered when directly working with the DEP force $\mathbf{f}_e = \mathbf{f}_e(\theta, \Phi)$ from (1.1) because it is a product of three gradients. For the typical regularity of solutions θ and Φ of heat equation and Gauss's law, respectively, on a Lipschitz domain with mixed boundary conditions and possible nonsmooth coefficients, it cannot be shown that $\mathbf{f}_e$ is an element of the underlying function space's dual. Therefore, $\mathbf{f}_e$ is replaced by a general force term $\mathbf{F}(\theta, \Phi)$ in (1.3). We first state assumptions on F that are sufficient for proving well-posedness of (1.3) and then propose certain ways for fitting the DEP force into the previously derived framework.

Concerning well-posedness, we note that for given θ and $\bar{\theta}$, the potential Φ is determined by Gauss's law. This fact motivates proving the existence of weak solutions of (1.3) by means of a fixed-point iteration. In this iteration, we split (1.3) into a hydrodynamical part, given by the stationary Boussinesq equations with buoyancy being replaced by $\mathbf{F}(\theta, \bar{\Phi})$ and Gauss' law with temperature-dependent permittivity.

The existence of solutions $(\mathbf{u}, \theta)$ of the Boussinesq equations is shown by employing a Galerkin principle combined with a fixed-point argument to a series of finite-dimensional problems. To be more precise, we adapt the concept presented in [14] for the stationary incompressible Navier Stokes equations. This result can be seen as generalization of [19], where existence and uniqueness of solutions of the stationary Boussinesq equations is shown for the standard buoyancy force.

The corresponding proof requires $\mathbf{F}$ to satisfy some type of weak continuity property. The proposed procedure demands stability of solutions $(\mathbf{u}, \theta)$ w.r.t. the data, measured in energy norm. This is established by combining ideas from the corresponding result for the stationary incompressible Navier Stokes equations (see e.g. [14]) with the procedure proposed in [19] and [18] to cope with non-homogenous Dirichlet boundary conditions for the temperature. We have to impose the assumption that there exists a family of smooth boundary liftings for θ_b of arbitrarily small L^3 norm. Moreover, we have to require the force term $\mathbf{F}$ to fulfill some boundedness principle of the form $\|\mathbf{F}(\theta, \Phi)\| \leq a_{\mathbf{F}}(\|\Phi\|)\|\nabla\theta\|$ for some non-decreasing function $a_{\mathbf{F}}$. Since the temperature determines the permittivity in Gauss' law, which has to be nonnegative, we need to ensure that θ is uniformly bounded. This result is obtained by means of a weak maximum principle and posing certain requirements on f_τ and θ_b.

Eventually, a uniqueness result for the stationary TEHD equations (1.3) under rather strict assumptions on the involved data is derived by modifying existing techniques and we obtain the requirement of $\mathbf{F}$ being locally Lipschitz continuous in some sense.

For the choice of $\mathbf{F}$, we suggest a model that is based on linearization around a smooth reference potential or that makes use of a regularization operator such as mollification. We give a heuristic justification for the proposed procedure in case of fluids with small permittivity variation γ and for small temperature variations.

The outline of this work is as follows: In Section 2 we formulate the Boussinesq problem with general force term $\mathbf{F}$ and summarize the requirements posed on $\mathbf{F}$. Afterward, we prove stability and existence of solutions and recall a maximum principle that is applicable to the temperature. In Section 3, we set up the variational formulation for the stationary TEHD equations and continue with proving existence, stability and uniqueness of solutions. As final step, we consider the modeling of the DEP force.

2 The Boussinesq Problem with General Force Term

After setting up the required notation, we first state a variational formulation for the stationary Boussinesq equations with general force term and collect results on existence and stability of its solutions. In Section 3, these results are needed to prove existence and uniqueness of solutions of (1.3).

Notation

Throughout this article, let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ denote a bounded and connected domain with Lipschitz boundary $\partial\Omega =: \Gamma =: \Gamma_D + \Gamma_N$ that is split into a Dirichlet and a Neumann part. Smooth functions are denoted by

$$\begin{aligned} C^{k,\alpha}(M) &:= \{v \in C^k(M) \text{ with } \alpha\text{-Hölder continuous derivatives up to order } k\} \text{ for } M \subset \mathbb{R}^d \\ C_0^\infty(\Omega) &:= \{v \in C^\infty(\Omega) :\ \operatorname{supp}(v) \subset\subset \Omega\} \\ C_D^{0,1}(\overline{\Omega}) &:= \{v \in C^{0,1}(\overline{\Omega}) :\ \operatorname{supp}(v) \cap \Gamma_D = \emptyset\} \end{aligned}$$

with $\operatorname{supp}(v) := \overline{\{x :\ v(x) \neq 0\}}$.

For $p \in [1, \infty]$ let $L^p(\Omega)$ denote the space of measurable and p-integrable functions on Ω with corresponding norm

$$\|u\|_p := \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \operatorname{esssup}_{x\in\Omega} |u(x)|, & p = \infty \end{cases}$$

Moreover, $L_0^p(\Omega) := \{v \in L^p(\Omega) :\ \int_\Omega v(x)dx = 0\}$ and $W^{k,p}(\Omega) := \{u \in L^p(\Omega) :\ \partial^\alpha u \in L^p(\Omega), |\alpha|_1 \leq k\}$ denotes the standard Sobolev space of integer order $k \geq 1$ with multiindex $\alpha \in \mathbb{N}_0^d$ and $|\alpha|_1 := \sum_{i=1}^d \alpha_i$.

The associated norms are defined as

$$\|u\|_{k,p} := \begin{cases} \left(\sum_{|\alpha|_1 \le k} \|\partial^\alpha u\|_p^p\right)^{\frac{1}{p}}, & p \in [1,\infty) \\ \max_{|\alpha|_1 \le k} \|\partial^\alpha u\|_\infty, & p = \infty \end{cases}.$$

In case of $p = 2$ we abbreviate $H^k(\Omega) := W^{k,2}(\Omega)$ with $W^{0,2}(\Omega) := L^2(\Omega)$ and $\|\cdot\| := \|\cdot\|_2$ and introduce the semi norm

$$|u|_k := \left(\sum_{|\alpha|_1 = k} \|\partial^\alpha u\|_2^2\right)^{\frac{1}{2}}.$$

For these Hilbert spaces, we denote their corresponding inner products as

$$(u,v)_k := \sum_{|\alpha|_1 \le k} (\partial^\alpha u, \partial^\alpha v), \text{ with } (u,v) := \int_\Omega u(x)v(x)dx.$$

For $p > 1$ let $\gamma_\Gamma \in \mathcal{L}(W^{1,p}(\Omega), W^{\frac{1}{p^*},p}(\Gamma))$ denote the trace operator and $\gamma_D \in \mathcal{L}(W^{1,p}(\Omega), W^{\frac{1}{p^*},p}(\Gamma_D))$ its restriction to the Dirichlet part of the boundary in case of nonzero $d-1$ dimensional Haussdorff measure of Γ_D. According to Theorem 1.5.1.3 in [12], there holds $\gamma_\Gamma v = v_{|\Gamma}$ and $\gamma_D v = v_{|\Gamma_D}$ for all Lipschitz continuous functions $v \in C^{0,1}(\overline{\Omega})$. Associated spaces are defined as

$$\begin{aligned} H_0^1(\Omega) &:= \overline{C_c^\infty(\Omega)}^{W^{1,2}} \subset \{v \in W^{1,2}(\Omega) : \gamma_\Gamma(v) = 0\} \\ H_D^1(\Omega) &:= \overline{C_D^{0,1}(\overline{\Omega}) \cap W^{1,6}(\Omega)}^{W^{1,2}} \subset \{v \in W^{1,2}(\Omega) : \gamma_D(v) = 0\}. \end{aligned}$$

Vector-valued function spaces are written in bold font, e.g. $\mathbf{L}^p(\Omega) := L^p(\Omega)^d$, and we use the same notation for the respective norms with $|\cdot|$ denoting the Euclidean norm.

Employing Friedrich's inequality A.7, Hölder's inequality and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we define constants for $p \in [1,p^*], q \ge p$ and $p^* \in [1,\infty]$ such that $-\frac{d}{p^*} \le 1 - \frac{d}{2}$

$$M_p := \sup_{\mathbf{v} \in \left(H_0^1(\Omega)\right)^d} \frac{\|\mathbf{v}\|_p}{\|\nabla \mathbf{v}\|},\ K_p := \sup_{\tau \in H_D^1(\Omega)} \frac{\|\tau\|_p}{\|\nabla \tau\|},\ K_{p,q} := \sup_{\tau \in L^q(\Omega)} \frac{\|\tau\|_p}{\|\tau\|_q}.$$

The positive and negative part of a function $u\colon \Omega \to \mathbb{R}$ are defined as $u^+(x) := \max\{u(x),0\}$ and $u^-(x) := \min\{u(x),0\}$, respectively. For $u,v \in H^1(\Omega)$ we define $u \le v$ on Γ_D iff $(u-v)^+ \in H_D^1(\Omega)$ and

$$\begin{aligned} \sup_{\Gamma_D} u &:= \inf\{k \in \mathbb{R} : u \le k \text{ on } \Gamma_D\} \\ \inf_{\Gamma_D} u &:= \sup\{k \in \mathbb{R} : k \le u \text{ on } \Gamma_D\}. \end{aligned}$$

There holds

$$\begin{aligned} \inf_{\Gamma_D} u = \sup\{k : (k-u)^+ \in H_D^1\} &= -\inf\{-k : (k-u)^+ \in H_D^1\} \\ &= -\inf\{k : (-k-u)^+ \in H_D^1\} = -\sup_{\Gamma_D}(-u). \end{aligned}$$

For a general normed space $(X, \|\cdot\|_X)$ its associated dual space is denoted by X^* with norm $\|\phi\|_{X^*} := \sup_{x \in X, \|x\|_X = 1} |\phi(x)|$. Moreover, for $x \in X, \phi \in X^*$ the dual pairing is denoted by $\phi(x) =: \langle \phi, x\rangle_{X^*,X} =: \langle \phi, x\rangle_{X^*}$. For $s > 0$ and $x \in X$, $B_s(x,X) := \{y \in X \ \ \|x-y\|_X < s\}$ denotes the open ball of radius s around x.

2.1 Variational Formulation

We define function spaces for velocity, $\mathbf{U} := \mathbf{H}_0^1(\Omega)$ and $\mathbf{V} = \{v \in \mathbf{U} : \nabla \cdot v = 0\}$, temperature, $\Theta := H_D^1(\Omega)$ and pressure, $M := L_0^2(\Omega)$. To shorten the following presentation of the Boussinesq equations, we introduce as set of bi- and trilinear forms given by

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) &:= \delta(\mathbf{u},\mathbf{v}) + \nu(\nabla\mathbf{u},\nabla\mathbf{v}), \quad c_v(\mathbf{u},\mathbf{v},w) := (\mathbf{u}\cdot\nabla\mathbf{v},w) \\ a_\tau(\theta,\tau) &:= \delta(\theta,\tau) + \kappa(\nabla\theta,\nabla\tau), \quad c_\tau(\mathbf{u},\theta,\tau) := (\mathbf{u}\cdot\nabla\theta,\tau) \\ b(\mathbf{u},q) &:= \frac{1}{\rho_r}(\nabla\cdot\mathbf{u},q). \end{aligned}$$

Lemma 2.1. *(Properties of trilinear forms)*

$$N_{\mathbf{v}} := \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{U}\setminus\{0\}} \frac{c_{\mathbf{v}}(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|} < \infty$$

$$N_\tau := \sup_{\mathbf{u}\in\mathbf{U}\setminus\{0\},\theta,\tau\in\Theta\setminus\{0\}} \frac{c_\tau(\mathbf{u},\theta,\tau)}{\|\nabla\mathbf{u}\|\|\nabla\theta\|\|\nabla\tau\|} < \infty$$

Moreover, if $\mathbf{u}\in\mathbf{V}$, then for $\mathbf{v},\mathbf{w}\in\mathbf{U}$, $\tau,\theta\in H^1(\Omega)$:

$$(i)\ c_{\mathbf{v}}(\mathbf{u},\mathbf{v},\mathbf{w}) = -c_{\mathbf{v}}(\mathbf{u},\mathbf{w},\mathbf{v}) \text{ and } (ii)\ c_\tau(\mathbf{u},\theta,\tau) = -c_\tau(\mathbf{u},\tau,\theta).$$

Proof. See Lemma II.1.1 and Lemma II.1.3 in [24]. □

By introducing an additional coefficient $\lambda\in[0,1]$, we obtain a family of stationary problems that will be used to prove existence by means of an appropriate fixed-point theorem. In the subsequent presentation, we will consider the problem in solenoidal form.

Problem 2.2. *(Stationary Boussinesq equations)*
Let $\theta_b\in C^1(\overline{\Omega})$ be a lifting of given boundary conditions and $\mathbf{F}\colon\Theta\to\mathbf{U}^*, \mathbf{f}_{\mathbf{v}}\in\mathbf{U}^*, f_\tau\in\Theta^*$ given body forces. Let either $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ denote fixed elements of $\mathbf{V}$ or the unknown velocity u. For $\lambda\in[0,1]$, $\delta\geq 0$, $\nu,\kappa>0$ find $\mathbf{u}\in\mathbf{V}$, $\theta\in\Theta$ such that

$$a_{\mathbf{v}}(\mathbf{u},\mathbf{v}) + \lambda\left(c_v(\bar{\mathbf{u}},\mathbf{u},\mathbf{v}) - \langle\mathbf{F}(\theta+\theta_b)+\mathbf{f}_{\mathbf{v}},\mathbf{v}\rangle_{\mathbf{U}^*}\right) = 0 \quad \forall\mathbf{v}\in\mathbf{V}$$

$$a_\tau(\theta,\tau) + \lambda\left(a_\tau(\theta_b,\tau) + c_\tau(\tilde{\mathbf{u}},\theta+\theta_b,\tau) - \langle f_\tau,\tau\rangle_{\Theta^*}\right) = 0 \quad \forall\tau\in\Theta$$

This problem can be written compactly in form of a fixed-point equation. To this end, we make use of a solution operator for the linear, elliptic part of Problem (2.2).

Definition 2.3. *(Stokes solution operator)*
Let $\mathbf{W}\subset\mathbf{U}$ and $T\subset\Theta$ be Hilbert spaces w.r.t. to the inner products $(\cdot,\cdot)_{\mathbf{W}} := (\nabla\cdot,\nabla\cdot)$ and $(\cdot,\cdot)_T := (\nabla\cdot,\nabla\cdot)$. Let $\delta\geq 0$, $\nu>0$, $\kappa>0$. The solution operator for the Stokes equations in solenoidal form combined with an additional heat equation, is defined as

$$\begin{aligned} K[\mathbf{W},T]\colon \mathbf{W}^*\times T^* &\to \mathbf{W}\times T \\ (f,g) &\mapsto (u,\theta) \text{ such that} \\ a_{\mathbf{v}}(u,v) &= -\langle f,v\rangle_{\mathbf{W}^*} \text{ for all } v\in\mathbf{W} \\ a_\tau(\theta,\tau) &= -\langle g,\tau\rangle_{T^*} \text{ for all } \tau\in T \end{aligned}$$

The remaining terms, including nonlinearities, source terms and boundary contributions, are collected in the operator N.

Definition 2.4. *(Non-Stokes terms)*
Let the assumptions of Problem 2.2 hold and let $\mathbf{W}\subset\mathbf{U}$ and $T\subset\Theta$ denote Hilbert spaces w.r.t. to the inner products $(\cdot,\cdot)_{\mathbf{W}} := (\nabla\cdot,\nabla\cdot)$ and $(\cdot,\cdot)_T := (\nabla\cdot,\nabla\cdot)$. Assume additionally that $\theta_b\in C^1(\overline{\Omega})$. We define

$$\begin{aligned} N[\mathbf{W},T]\colon \mathbf{W}\times T &\to \mathbf{W}^*\times T^* \\ (\mathbf{u},\theta) &\mapsto \begin{pmatrix} c_v(\bar{\mathbf{u}},\mathbf{u},\cdot) - \mathbf{F}(\theta+\theta_b) - \mathbf{f}_{\mathbf{v}} \\ a_\tau(\theta_b,\cdot) + c_\tau(\tilde{\mathbf{u}},\theta+\theta_b,\cdot) - f_\tau \end{pmatrix} \end{aligned}$$

Problem 2.5. *(Fixed-point version)*
Let the assertions of Problem 2.2, Definition 2.3 and Definition 2.4 hold. For $K := K[\mathbf{W}, T]$, $N := N[\mathbf{W}, T]$ *and* $\lambda \in [0, 1]$ *find* $(\mathbf{u}_\lambda, \theta_\lambda) \in \mathbf{W} \times T$ *such that*

$$(\mathbf{u}_\lambda, \theta_\lambda) = \lambda K(N(\mathbf{u}_\lambda, \theta_\lambda)) = \lambda \mathcal{F}(\mathbf{u}_\lambda, \theta_\lambda). \tag{2.1}$$

with fixed-point operator

$$\mathcal{F} := \mathcal{F}[\mathbf{W}, T] := K \circ N := : \mathbf{W} \times T \to \mathbf{W} \times T.$$

In order to apply an appropriate existence theorem to the Problem 2.5, we need to show compactness of $\mathcal{F}$. For finite dimensional spaces, this can be accomplished by the next lemmas below.

Lemma 2.6. *(Properties of* K*)*
Let the assumptions of Definition 2.3 hold and define $K := K[\mathbf{W}, T]$*. Then, he following assertions hold.*

- *(i)* K *is well defined and linear*
- *(ii) There exists* $C_K \geq 0$ *such that* $(u, \theta) = K(f, g)$ *satisfies* $\|u\|_{\mathbf{W}} + \|\theta\|_T \leq C_K(\|f\|_{\mathbf{W}^*} + \|g\|_{T^*})$

Proof. Follows by application of Lax-Milgram A.3 and Friedrich's inequalities A.5, A.7. □

Lemma 2.7. *(Properties of* N*)*
Let the assumptions of Problem 2.2 and Definition 2.4 hold. Assume that $\mathbf{F}(\cdot + \theta_b): T \to \mathbf{W}^*$ *is continuous. Then,* $N := N[\mathbf{W}, T]$ *is continuous w.r.t. the norms* $\|\cdot\|_{\mathbf{W} \times T} := \|\nabla \cdot\| + \|\nabla \cdot\|$ *and* $\|\cdot\|_{\mathbf{W}^* \times T^*} := \|\cdot\|_{\mathbf{W}^*} + \|\cdot\|_{T^*}$.

Proof. We only show the proof for $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = \mathbf{u}$. Continuity of $\mathbf{W} \ni u \mapsto c_v(\mathbf{u}, \mathbf{u}, \cdot) \in \mathbf{W}^*$ follows directly from the following estimation

$$\begin{aligned}
|c_v(\mathbf{u}, \mathbf{u}, v) - c_v(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v})| &= |c_v(\mathbf{u} - \mathbf{u}_n, u, \mathbf{v}) + c_v(\mathbf{u}_n, \mathbf{u} - \mathbf{u}_n, \mathbf{v})| \\
&\leq N_{\mathbf{v}} \|\nabla(\mathbf{u} - \mathbf{u}_n)\| \|\nabla \mathbf{u}\| \|\nabla v\| + N_{\mathbf{v}} \|\nabla \mathbf{u}_n\| \|\nabla(\mathbf{u} - \mathbf{u}_n)\| \|\nabla \mathbf{v}\| \\
&= N_{\mathbf{v}} \left(\underbrace{\|\nabla(\mathbf{u} - \mathbf{u}_n)\|}_{\to 0} \|\nabla \mathbf{u}\| + \underbrace{\|\nabla \mathbf{u}_n\|}_{\to \|\nabla \mathbf{u}\|} \underbrace{\|\nabla(\mathbf{u} - \mathbf{u}_n)\|}_{\to 0} \right) \|\nabla \mathbf{v}\|
\end{aligned}$$

for $(\mathbf{u}, \theta) \in \mathbf{W} \times T$ and $(\mathbf{u}_n, \theta_n)_n \subset \mathbf{W} \times T$ with $\|\nabla(\mathbf{u} - \mathbf{u}_n)\| \to 0$ and $\|\nabla(\theta - \theta_n)\| \to 0$.

Analogously, $\mathbf{W} \times T \ni (\mathbf{u}, \theta) \mapsto c_\tau(\mathbf{u}, \theta, \cdot) \in T^*$ is continuous. By continuity of $\mathbf{F}$, N is composition of continuous functions and therefore continuous as mapping from $(\mathbf{W} \times T, \|\cdot\|_{\mathbf{W} \times T})$ to $(\mathbf{W}^* \times T^*, \|\cdot\|_{\mathbf{W}^* \times T^*})$
. □

Lemma 2.8. *(Properties of* $\mathcal{F}$*)*
Assume that $\theta_b \in C^1(\overline{\Omega})$ *with* $\nabla \theta_b \cdot \mathbf{n} = 0$ *on* Γ_N*. Moreover, let the assertions in Problem 2.5 and of Lemma 2.6 and 2.7 hold. Then, Problem 2.2 and 2.5 are equivalent if* $\mathbf{W} \times T = \mathbf{V} \times \Theta$*. Moreover, the following properties of the fixed-point operator* $\mathcal{F}[U, T]$ *hold:*

- *(i)* $\mathcal{F}[\mathbf{W}, T]$ *is continuous*
- *(ii)* $\mathcal{F}[\mathbf{W}, T]$ *is compact, i.e. it maps bounded sets to sets with compact closure, if* $\mathbf{W}$ *and* T *are finite dimensional.*

Proof. (i) follows since $\mathcal{F}$ is a composition of continuous functions due to Lemma 2.6 and 2.7. If $\mathbf{W}$ and T are finite dimensional, (ii) follows since $\mathcal{F}$ is a continuous map on a finite dimensional Hilbert space, see Lemma A.2. To see the equivalence of Problem 2.2 and 2.5, let $(u_\lambda, \theta_\lambda)$ denote a solution of Problem 2.5. By definition of K, this solution satisfies

$$\begin{pmatrix} a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) \\ a_\tau(\theta, \tau) \end{pmatrix} = -\lambda N(\mathbf{u}_\lambda, \theta_\lambda)(\mathbf{v}, \tau) \text{ for all } (\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$$

Inserting the definition of N yields the assertion. □

We conclude this section with stating the assumptions on $\mathbf{F}$ which turn out to be sufficient for proving certain well-posedness results for the generalized Boussinesq equations.

Assumption 2.9. *(General body force)*
Let $\mathbf{F}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbf{U}^*$ *satisfy*

(i) $\mathbf{F}$ *is locally Lipschitz continuous in the following sense: for all* $R > 0$ *there is a non-decreasing function* $L_{\mathbf{F}}^{(\theta)}\colon [0,\infty) \to [0,\infty)$ *such that*

$$|\langle \mathbf{F}(\theta_1, \Phi) - \mathbf{F}(\theta_2, \Phi), \mathbf{v}\rangle_{\mathbf{U}^*}| \le L_{\mathbf{F}}^{(\theta)}(\|\Phi\|_{1,2})\|\theta_1 - \theta_2\|_{1,2}\|\nabla \mathbf{v}\|$$

for all $\theta_1, \theta_2 \in B_R(0, H^1(\Omega))$, $\Phi \in H^1(\Omega)$ *and* $\mathbf{v} \in \mathbf{U}$.
Moreover, for all $R > 0$ *there is* $L_{\mathbf{F}}^{(\Phi)} \ge 0$ *such that for all* $\theta \in H^1(\Omega)$, $\Phi_1, \Phi_2 \in B_R(0, H^1(\Omega))$ *and* $\mathbf{v} \in \mathbf{U}$,

$$|\langle \mathbf{F}(\theta, \Phi_1) - \mathbf{F}(\theta, \Phi_2), \mathbf{v}\rangle_{\mathbf{U}^*}| \le L_{\mathbf{F}}^{(\Phi)}\|\theta\|_{1,2}\|\Phi_1 - \Phi_2\|_{1,2}\|\nabla \mathbf{v}\|.$$

(ii) $\mathbf{F}$ *is bounded in the following sense: There are non-decreasing functions*

$$a_{\mathbf{F}}\colon [0,\infty) \to [0,\infty) \quad \text{and} \quad b_{\mathbf{F}}\colon [0,\infty) \to [0,\infty),$$

such that

$$|\langle \mathbf{F}(\theta, \Phi), \mathbf{v}\rangle_{\mathbf{U}^*}| \le a_{\mathbf{F}}(\|\Phi\|_{1,2})\|\theta\|_{1,2}\|\nabla \mathbf{v}\| + b_{\mathbf{F}}(\|\Phi\|_{1,2})\|\nabla \mathbf{v}\|$$

for all $\theta \in H^1(\Omega), \Phi \in H^1(\Omega), \mathbf{v} \in \mathbf{U}$.

(iii) *Let sequences* $(\theta_n)_n \subset H^1(\Omega)$ *and* $(\Phi_n)_n \subset H^1(\Omega)$ *be given that converge to* $\theta_* \in H^1(\Omega)$ *and* $\Phi_* \in H^1(\Omega)$, *respectively, in the following sense*

$$\theta_n \rightharpoonup \theta_* \text{ in } H^1,\ \theta_n \to \theta_* \text{ in } L^4 \text{ and } \|\theta_n\|_{1,2} \le K \text{ for all } n \in \mathbb{N}$$
$$\Phi_n \rightharpoonup \Phi_* \text{ in } H^1, \Phi_n \to \Phi_* \text{ in } L^4 \text{ and } \|\Phi_n\|_{1,2} \le K \text{ for all } n \in \mathbb{N}.$$

Then, for all $\mathbf{v} \in \mathbf{U}$ *there holds true*

$$|\langle \mathbf{F}(\theta_*, \Phi_*) - \mathbf{F}(\theta_n, \Phi_n), \mathbf{v}\rangle_{\mathbf{U}^*}| \to 0.$$

The local Lipschitz condition (i) is used to show uniqueness of solutions under an appropriate small data condition. It will turn out that Assumption (ii) allows for showing stability of solutions to Problem 2.2. Note that we don't restrict the growth rate of $\mathbf{F}$ w.r.t. $\|\Phi\|_{1,2}$, while it may grow at most linearly w.r.t. $\|\theta\|_{1,2}$. This is due to the fact that we will be able to bound $\|\Phi\|_{1,2}$ in terms of $\|\theta\|_\infty$ which, in turn, can be bounded in terms of the input data only by virtue of a maximum principle. (iii) will be needed to show that a sequence of solutions to finite dimensional versions of the fixed-point Problem 2.5 converges to a solution of the original Problem 2.2.

Throughout the subsequent sections, we assert the following main assumption concerning $\mathbf{F}$ and the boundary liftings.

Assumption 2.10. *(Boundary Lifting)*
There exists a family of boundary liftings $\theta_b = \theta_b[\xi] \in C^1(\overline{\Omega})$ *with* $\|\theta_b[\xi]\|_3 \le \xi$ *for all* $\xi > 0$.

2.2 Stability of Solutions

The following proposition states that the norm of solutions of Problem (2.2) can be bounded in terms of the input parameters and the norm of the boundary lifting. For that reason, it is required that the boundary lifting θ_b can be chosen in such a way that $\|\theta_b\|_3$ is sufficiently small.

The subsequent result will play an important role in showing both existence (by virtue of a fixed-point argument) and uniqueness of solutions. For the former case, it is crucial that the stability bound is uniform w.r.t. $\lambda \in [0,1]$.

Proposition 2.11. *(Stability of stationary solutions)*
Let $\lambda \in [0,1]$ and assume that Assumptions 2.9 and 2.10 holds with body force $\mathbf{F}(\theta) := \mathbf{F}(\theta, \Phi)$ defined for given and fixed $\Phi \in H^1(\Omega)$ with respective constants $a_{\mathbf{F}} := a_{\mathbf{F}}(\|\Phi\|_{1,2}), b_{\mathbf{F}} := b_{\mathbf{F}}(\|\Phi\|_{1,2})$.

Then, there is a continuous, non-increasing function $d\colon [0,\infty) \to (0, \frac{1}{2}]$ with $d(0) > 0$ and continuous functions

$$g_i\colon \mathbb{R}^6 \cap \{x_1 < d(x_2)\} \to [0,\infty),\ i \in \{\mathbf{u}, \theta\}$$

such that

$$\|\nabla \mathbf{u}\| \le g_{\mathbf{u}}(\|\theta_b\|_3, a_{\mathbf{F}}, b_{\mathbf{F}}, \|\theta_b\|_{1,2}, \|f_\tau\|_{\Theta^*}, \|\mathbf{f_v}\|_{\mathbf{U}^*}) =: G_{\mathbf{u}}$$
$$\|\nabla \theta\| \le g_{\theta}(\|\theta_b\|_3, a_{\mathbf{F}}, b_{\mathbf{F}}, \|\theta_b\|_{1,2}, \|f_\tau\|_{\Theta^*}, \|\mathbf{f_v}\|_{\mathbf{U}^*}) =: G_{\theta}$$

for all solutions $(\mathbf{u}, \theta)$ of Problem 2.2 with boundary lifting θ_b satisfying $\|\theta_b\|_3 < d(a_{\mathbf{F}})$. In particular, $g_{\mathbf{u}}$ and g_θ do not depend on the parameter $\lambda \in [0,1]$ and are non-decreasing in their arguments x_2 and x_3. Moreover, if $\tilde{\mathbf{u}} = \mathbf{u}$, then

$$g_i \to 0 \text{ for } (x_4, x_5, x_6) \to 0, \ \text{ i.e. } G_i \to 0 \text{ for } (\|\theta_b\|_{1,2}, \|f_\tau\|, \|\mathbf{f_v}\|) \to 0.$$

Proof. Let $\theta_b \in C^1(\overline{\Omega})$ such that $\|\theta_b\|_3 \le 1$. Moreover, let $(\mathbf{u}, \theta) \in \mathbf{V} \times \Theta$ denote a solution of the stationary problem. Inserting $\mathbf{v} = \mathbf{u}, \tau = \theta$ in 2.2 and noting that

$$c_{\mathbf{v}}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{u}) = 0 \text{ and } c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \theta) = c_\tau(\tilde{\mathbf{u}}, \theta_b, \theta) = -c_\tau(\tilde{\mathbf{u}}, \theta, \theta_b)$$

since $\bar{\mathbf{u}}, \tilde{\mathbf{u}} \in \mathbf{V}$, we obtain

$$\delta\|\mathbf{u}\|^2 + \nu\|\nabla \mathbf{u}\|^2 = \lambda\langle \mathbf{F}(\theta + \theta_b) + \mathbf{f_v}, u\rangle$$
$$\delta\|\theta\|^2 + \kappa\|\nabla\theta\|^2 = -\lambda\left(\delta(\theta_b, \theta) + \kappa(\nabla\theta_b, \nabla\theta) + (\tilde{\mathbf{u}} \cdot \nabla\theta, \theta_b) - \langle f_\tau, \theta\rangle\right).$$

Using the assumptions on $\mathbf{F}$, we obtain from the first equality

$$\begin{aligned}\nu\|\nabla\mathbf{u}\|^2 &\le \underbrace{a_{\mathbf{F}}\sqrt{K_2^2+1}}_{\tilde{a}_{\mathbf{F}}}\|\nabla\theta\|\|\nabla\mathbf{u}\| + \underbrace{(b_{\mathbf{F}} + a_{\mathbf{F}}\|\theta_b\|_{1,2} + \|\mathbf{f_v}\|_{\mathbf{U}^*})}_{=:\tilde{b}_{\mathbf{F}}}\|\nabla\mathbf{u}\| \\ &\le \tilde{a}_{\mathbf{F}}\left(\frac{1}{2\delta_1}\|\nabla\theta\|^2 + \frac{\delta_1}{2}\|\nabla\mathbf{u}\|^2\right) + \frac{1}{2\delta_2}\tilde{b}_{\mathbf{F}}^2 + \frac{\delta_2}{2}\|\nabla\mathbf{u}\|^2\end{aligned}$$

for $\delta_1, \delta_2 > 0$ and from the second one

$$\begin{aligned}\kappa\|\nabla\theta\|^2 &\le \kappa\|\nabla\theta_b\|\|\nabla\theta\| + \delta K_2\|\theta_b\|\|\nabla\theta\| + |(\tilde{\mathbf{u}}\cdot\nabla\theta, \theta_b)| + \|f_\tau\|_{\Theta^*}\|\nabla\theta\| \\ &\le \kappa\|\nabla\theta_b\|\|\nabla\theta\| + \delta K_2 K_{23}\|\theta_b\|_3\|\nabla\theta\| + M_6\|\nabla\tilde{\mathbf{u}}\|\|\nabla\theta\|\|\theta_b\|_3 + \|f_\tau\|_{\Theta^*}\|\nabla\theta\| \\ &\le \kappa\left(\frac{1}{2\delta_3}\|\nabla\theta_b\|^2 + \frac{\delta_3}{2}\|\nabla\theta\|^2\right) + M_6\left(\frac{1}{2\delta_4}\|\theta_b\|_3\|\nabla\tilde{\mathbf{u}}\|^2 + \frac{\delta_4}{2}\|\theta_b\|_3\|\nabla\theta\|^2\right) \\ &+ \left(\frac{1}{2\delta_5}\|f_\tau\|_{\Theta^*}^2 + \frac{\delta_5}{2}\|\nabla\theta\|^2\right) + \delta K_2 K_{23}\left(\frac{1}{2\delta_6}\|\theta_b\|_3^2 + \frac{\delta_6}{2}\|\nabla\theta\|^2\right)\end{aligned}$$

for $\delta_3, \delta_4, \delta_5, \delta_6 > 0$. Rearranging terms yields

$$(\nu - \frac{\delta_1 \tilde{a}_{\mathbf{F}}}{2} - \frac{\delta_2}{2})\|\nabla \mathbf{u}\|^2 \leq \tilde{a}_{\mathbf{F}} \frac{1}{2\delta_1}\|\nabla \theta\|^2 + \frac{1}{2\delta_2}\tilde{b}_{\mathbf{F}}^2$$

$$(\kappa - \tilde{\kappa}\frac{\delta_3}{2} - M_6\|\theta_b\|_3\frac{\delta_4}{2} - \frac{\delta_5}{2} - \delta K_2 K_{23}\frac{\delta_6}{2})\|\nabla \theta\|^2 \leq \kappa \frac{1}{2\delta_3}\|\nabla \theta_b\|^2 + M_6 \frac{1}{2\delta_4}\|\theta_b\|_3\|\nabla \tilde{\mathbf{u}}\|^2 + \frac{1}{2\delta_5}\|f_\tau\|_{\Theta^*}^2 + \delta K_2 K_{23}\frac{1}{2\delta_6}\|\theta_b\|_3^2.$$

Setting δ_i appropriately gives

$$\|\nabla \mathbf{u}\|^2 \leq C_1\|\nabla \theta\|^2 + C_2$$

$$(1 - 2\|\theta_b\|_3)\|\nabla \theta\|^2 \leq C_3\|\nabla \theta_b\|^2 + C_4\|\theta_b\|_3\|\nabla \tilde{\mathbf{u}}\|^2 + C_5 + C_6\|\theta_b\|_3^2$$

with constants

$$C_1 = \frac{2}{\nu^2}a_{\mathbf{F}}^2(K_2^2 + 1),\ C_2 = \frac{2}{\nu^2}\left(b_{\mathbf{F}} + a_{\mathbf{F}}\|\theta_b\|_{1,2} + \|\mathbf{f_v}\|_{\mathbf{U}^*}\right)^2,\ C_3 = 3$$

$$C_4 = \frac{M_6^2}{2\kappa^2},\ C_5 = \frac{3}{\kappa^2}\|f_\tau\|_{\Theta^*}^2,\ C_6 = 3\left(\frac{\delta K_2 K_{23}}{\kappa}\right)^2.$$

If $\tilde{\mathbf{u}} = \mathbf{u}$, we set

$$d := d(a_{\mathbf{F}}) := \min\left(\frac{1}{2 + C_1 C_4}, \frac{1}{2}\right),$$

and obtain for $\|\theta_b\|_3 < d$ by combination of both inequalities

$$\|\nabla \mathbf{u}\|^2 \leq \frac{1}{1 - C_1 C_4 \frac{\|\theta_b\|_3}{1-2\|\theta_b\|_3}}\left(C_2 + \frac{C_1}{1 - 2\|\theta_b\|_3}\left(C_3\|\nabla \theta_b\|^2 + C_5 + C_6\|\theta_b\|_3^2\right)\right) =: G_{\mathbf{u}}^2.$$

Now, θ can be bounded by

$$\|\nabla \theta\| \leq \frac{1}{1 - 2\|\theta_b\|_3}\left(C_3\|\nabla \theta_b\|^2 + C_4 G_{\mathbf{u}}^2 + C_5 + C_6\|\theta_b\|_3^2\right) =: G_\theta^2.$$

If $\tilde{\mathbf{u}}$ is fixed, we get for $\|\theta_b\|_3 < \frac{1}{2}$:

$$\|\nabla \theta\|^2 \leq \frac{1}{1 - 2\|\theta_b\|_3}\left(C_3\|\nabla \theta_b\|^2 + C_4\|\nabla \tilde{\mathbf{u}}\|^2 + C_5 + C_6\|\theta_b\|_3^2\right) =: G_\theta^2$$

$$\|\nabla \mathbf{u}\|^2 \leq C_1 G_\theta^2 + C_2 =: G_{\mathbf{u}}^2.$$

□

Remark 2.12. The previous proof shows that Proposition 2.11 is valid for arbitrary subspaces $\mathbf{W} \times T \subset \mathbf{V} \times \Theta$ with functions $d, g_{\mathbf{u}}, g_\theta$ that are independent of the specific choice of $\mathbf{W} \times T$.

The next lemma bounds the variation of solutions of the Boussinesq equations w.r.t. to variations of the respective input data, i.e. the force term $\mathbf{F}$ and the terms $\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}$. These bounds will be used later on for showing uniqueness of solutions.

Lemma 2.13. *(Stability of stationary solution w.r.t varying data)*
Let Assumptions 2.9 and 2.10 hold and denote $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ solutions of Problem 2.2 for $\lambda = 1$, respective convection fields $(\bar{\mathbf{u}}^{(1)}, \tilde{\mathbf{u}}^{(1)})$, $(\bar{\mathbf{u}}^{(2)}, \tilde{\mathbf{u}}^{(2)})$ and external forces $\mathbf{F}_1 = \mathbf{F}_1(\cdot, \Phi_1)$, $\mathbf{F}_2 = \mathbf{F}_2(\cdot, \Phi_2)$ which do both satisfy Assumption 2.9. Assume

$$D_{\mathbf{F}} := \sup_{w \in \mathbf{V}} \sup_{\theta \in \Theta} \frac{|\langle \mathbf{F}_1(\theta + \theta_b) - \mathbf{F}_2(\theta + \theta_b), w\rangle_{\mathbf{U}^*}|}{\|\nabla w\|\|\theta + \theta_b\|_{1,2}} < \infty.$$

Then, the solutions satisfy

$$\begin{aligned}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\| &\le D_1\|\nabla(\bar{\mathbf{u}}^{(1)}-\bar{\mathbf{u}}^{(2)})\| + D_2\|\nabla(\tilde{\mathbf{u}}^{(1)}-\tilde{\mathbf{u}}^{(2)})\| + D_3 D_{\mathbf{F}}\\ \|\nabla(\theta_1-\theta_2)\| &\le D_4\|\nabla(\tilde{\mathbf{u}}^{(1)}-\tilde{\mathbf{u}}^{(2)})\|.\end{aligned}$$

with constants given by (2.2).

Proof. Let $(\mathbf{u}_1,\theta_1)$ and $(\mathbf{u}_2,\theta_2)$ denote two solutions for external forces $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. Inserting both into 2.2 and subtracting yields

$$\begin{aligned}a_{\mathbf{v}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{v}) + (\bar{\mathbf{u}}^{(1)}\cdot\nabla\mathbf{u}_1,\mathbf{v}) - (\bar{\mathbf{u}}^{(2)}\cdot\nabla\mathbf{u}_2,\mathbf{v}) &= \langle \mathbf{F}_1(\theta_1+\theta_b) - \mathbf{F}_2(\theta_2+\theta_b),\mathbf{v}\rangle \;\forall \mathbf{v}\in\mathbf{V}\\ a_\tau(\theta_1-\theta_2,\tau) + (\tilde{\mathbf{u}}^{(1)}\cdot\nabla\theta_1,\tau) - (\tilde{\mathbf{u}}^{(2)}\cdot\nabla\theta_2,\tau) &= -((\tilde{\mathbf{u}}^{(1)}-\tilde{\mathbf{u}}^{(2)})\cdot\nabla\theta_b,\tau)\;\forall\tau\in\Theta\end{aligned}$$

Defining $w:=\mathbf{u}_1-\mathbf{u}_2,\ \bar{\mathbf{u}}:=\bar{\mathbf{u}}^{(1)}-\bar{\mathbf{u}}^{(2)},\ \tilde{\mathbf{u}}:=\tilde{\mathbf{u}}^{(1)}-\tilde{\mathbf{u}}^{(2)},\ \phi:=\theta_1-\theta_2$ and setting $\mathbf{v}=w,\ \tau=\phi$ yields

$$\begin{aligned}\delta\|w\|^2+\nu\|\nabla w\|^2 + (\bar{\mathbf{u}}\cdot\nabla\mathbf{u}_1,w) + \underbrace{(\bar{\mathbf{u}}^{(2)}\cdot\nabla w,w)}_{=0} &= \langle\mathbf{F}_1(\theta_1+\theta_b)-\mathbf{F}_2(\theta_2+\theta_b),w\rangle\\ \delta\|\phi\|^2+\kappa\|\nabla\phi\|^2 + (\tilde{\mathbf{u}}\cdot\nabla\theta_1,\phi) + \underbrace{(\tilde{\mathbf{u}}^{(2)}\cdot\nabla\phi,\phi)}_{=0} &= -(\tilde{\mathbf{u}}\cdot\nabla\theta_b,\phi) = (\tilde{\mathbf{u}}\cdot\nabla\phi,\theta_b)\end{aligned}$$

Let $R:=\|\theta_b\|_{1,2}+\sqrt{K_2^2+1}G_\theta^{(1)}$ and by using local Lipschitz continuity of $\mathbf{F}_1$ with constant $L_{\mathbf{F}_1}^{(\theta)}=L_{\mathbf{F}_1}^{(\theta)}(R)$

$$\begin{aligned}\langle\mathbf{F}_1(\theta_1+\theta_b)-\mathbf{F}_2(\theta_2+\theta_b),w\rangle &= \langle\mathbf{F}_1(\theta_1+\theta_b)-\mathbf{F}_1(\theta_2+\theta_b),w\rangle + \langle\mathbf{F}_1(\theta_2+\theta_b)-\mathbf{F}_2(\theta_2+\theta_b),w\rangle\\ &\le L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2+1}\|\nabla(\theta_1-\theta_2)\|\|\nabla w\| + D_{\mathbf{F}}\|\theta_2+\theta_b\|_{1,2}\|\nabla w\|,\end{aligned}$$

we obtain

$$\begin{aligned}\nu\|\nabla w\|^2 &\le N_{\mathbf{v}}\|\nabla\bar{\mathbf{u}}\|\|\nabla w\|\|\nabla\mathbf{u}_1\| + L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2+1}\|\nabla(\theta_1-\theta_2)\|\|\nabla w\| + D_{\mathbf{F}}\|\theta_2+\theta_b\|_{1,2}\|\nabla w\|\\ &\le N_{\mathbf{v}}G_{\mathbf{u}}^{(1)}\|\nabla\bar{\mathbf{u}}\|\|\nabla w\| + L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2+1}\|\nabla\phi\|\|\nabla w\| + D_{\mathbf{F}}\left(G_\theta^{(2)}\sqrt{K_2^2+1}+\|\theta_b\|_{1,2}\right)\|\nabla w\|\end{aligned}$$

and from the second equation,

$$\begin{aligned}\kappa\|\nabla\phi\|^2 &\le N_\tau\|\nabla\tilde{\mathbf{u}}\|\|\nabla\theta_1\|\|\nabla\phi\| + M_6\|\nabla\tilde{\mathbf{u}}\|\|\nabla\phi\|\|\theta_b\|_3\\ &\le N_\tau G_\theta^{(1)}\|\nabla\tilde{\mathbf{u}}\|\|\nabla\phi\| + M_6\|\nabla\tilde{\mathbf{u}}\|\|\nabla\phi\|\|\theta_b\|_3\end{aligned}$$

Dividing by $\|\nabla w\|$ and $\|\nabla\phi\|$, respectively, yields the desired result with constants

$$\begin{aligned}D_1 &= \frac{1}{\nu}N_{\mathbf{v}}G_{\mathbf{u}}^{(1)}\\ D_2 &= \frac{1}{\nu\kappa}L_{\mathbf{F}_1}^{(\theta)}\sqrt{K_2^2+1}(N_\tau G_\theta^{(1)}+M_6\|\theta_b\|_3)\\ D_3 &= \frac{1}{\nu}\left(G_\theta^{(2)}\sqrt{KS_2^2+1}+\|\theta_b\|_{1,2}\right)\\ D_4 &= \frac{1}{\kappa}(N_\tau G_\theta^{(1)}+M_6\|\theta_b\|_3).\end{aligned}\tag{2.2}$$

□

2.3 Weak Maximum Principle for Temperature

If the dielectirc permittivity ϵ is chosen as linear function of the temperature, e.g. $\epsilon(\theta) = \epsilon_r(1-\gamma\theta)$ as done in [20], we need to provide a L^∞ bound on θ. This bound is obtained by means of the well-known weak maximum principle, stating that θ exhibits its extremal values on the boundary. The following fundamental functional analytic result will be frequently used.

Theorem 2.14. *(Continuity of Superposition Operator, Theorem 1 in [16])*
Let $f \in C^{0,1}(\mathbb{R})$ and G denote a bounded domain. Then the following operator is continuous

$$T_f : W^{1,p}(G) \to W^{1,p}(G)$$
$$v \mapsto f \circ v$$

By setting $f(s) := \max\{s, 0\}$ or $f(s) := \min\{s, 0\}$, Theorem 2.14 shows that $u^\pm \in W^{1,2}(\Omega)$ for $u \in W^{1,2}(\Omega)$.

Proposition 2.15. *(Maximum principle for θ)*
Let $(\mathbf{u}, \theta) \in \mathbf{U} \times \Theta$ denote a solution of Problem 2.2 for $\lambda = 1$ and assume that one of the following conditions is satisfied for all $\tau \in \Theta$.

(i) $\delta > 0$ *and* $|\langle f_\tau, \tau\rangle_{\Theta^*}| \le \kappa\|\nabla\tau\|^2$

(ii) $|\langle f_\tau, \tau\rangle_{\Theta^*}| \le \delta\|\tau\|^2$

Then, $\theta + \theta_b$ exhibits its maximum and minimum on the boundary, i.e.

$$\operatorname{esssup}_\Omega(\theta + \theta_b) \le \sup_{\Gamma_D} \theta_b^+ \text{ and } \operatorname{essinf}_\Omega(\theta + \theta_b) \ge \inf_{\Gamma_D} \theta_b^- .$$

Proof. Let $\overline{\theta} := \theta + \theta_b$. Then,

$$\delta(\overline{\theta}, \tau) + \kappa(\nabla\overline{\theta}, \nabla\tau) + c_\tau(\tilde{\mathbf{u}}, \overline{\theta}, \tau) = \langle f_\tau, \tau\rangle \text{ for all } \tau \in \Theta. \tag{2.3}$$

For $v \in H^1(\Omega)$, let $S(v) := \{k \in \mathbb{R} : \ v^+ \le k \text{ on } \Gamma_D\}$, i.e. $\sup_{\Gamma_D} v^+ = \inf S(v)$. We assume that $S(\theta_b) \ne \emptyset$. Otherwise, $\sup_{\Gamma_D} \theta_b^+ = \inf \emptyset := \infty$ and the assertion trivially holds.

Step (i) : $\sup_{\Gamma_D}(\overline{\theta}^+) \le \sup_{\Gamma_D}(\theta_b^+)$.
We prove that $S(\theta_b) \subset S(\overline{\theta})$. To do so, let $k \in S(\theta_b)$ and define $f(s) := (s^+ - k)^+$ which is Lipschitz continuous on $\mathbb{R}$. Since $k \in S(\theta_b)$, there holds $f(\theta_b) \in H^1_D(\Omega)$ and

$$f(\overline{\theta}) - f(\theta_b) = ((\theta + \theta_b)^+ - k)^+ - (\theta_b^+ - k)^+ =: z \in H^1(\Omega) \text{ according to Theorem 2.14.}$$

Since $\theta \in H^1_D(\Omega)$, there is $(\theta_n)_n \subset C^{0,1}_D(\Omega) \cap W^{1,6}(\Omega)$ with $\theta_n \to \theta$ in $H^1(\Omega)$. Let

$$z_n := f(\theta_n + \theta_b) - f(\theta_b).$$

Since $f \in C^{0,1}(\mathbb{R})$, $\theta_n \in C^{0,1}_D(\overline{\Omega})$ and $\theta_b \in C^1(\overline{\Omega})$, there holds $z_n \in C^{0,1}(\overline{\Omega})$. Moreover, using $\theta_n, \theta_b \in W^{1,6}(\Omega)$ and Theorem 2.14, we obtain $z_n \in W^{1,6}(\Omega)$. Finally, for $x \notin \operatorname{supp}(\theta_n)$, i.e. $\theta_n(x) = 0$, we also have $z_n(x) = 0$. Therefore, $\operatorname{supp}(z_n) \subset \operatorname{supp}(\theta_n)$ which implies $z_n \in C^{0,1}_D(\overline{\Omega})$. By definition of z_n, θ_n and Theorem 2.14,

$$z - z_n = f(\theta + \theta_b) - f(\theta_n + \theta_b) \to 0 \text{ in } H^1(\Omega), \text{ since } \theta_n \to \theta \text{ in } H^1(\Omega).$$

Thus, $z \in H^1_D(\Omega)$ and $f(\overline{\theta}) = f(\theta_b) + z \in H^1_D(\Omega)$, implying $k \in S(\overline{\theta})$.

Step (ii) : $\text{esssup}_\Omega(\overline{\theta}) \leq \sup_{\Gamma_D}(\overline{\theta}^+)$.
Suppose that $\text{esssup}_\Omega(\overline{\theta}) > \sup_{\Gamma_D}(\overline{\theta}^+) = \inf S(\overline{\theta})$. According to step (i), $\emptyset \neq S(\theta_b) \subset S(\overline{\theta})$. Thus, there exists $k \geq 0$ with $k \in S(\overline{\theta})$ and $\text{esssup}_\Omega(\overline{\theta}) > k$.

By definition of k, $\tau := (\overline{\theta}^+ - k)^+ \in H^1_D(\Omega)$ with support $E := \{\tau \neq 0\} = \{\overline{\theta}^+ > k\} = \{\overline{\theta} > k\}$. Moreover, there holds

$$\tau = \overline{\theta} - k \text{ on } E \text{ and } \nabla\tau = \begin{cases} \nabla\overline{\theta} \text{ on } E \\ 0 \text{ else} \end{cases}.$$

Inserting this τ as test function into (2.3) yields

$$\begin{aligned}\langle f_\tau, \tau\rangle = \int_\Omega \delta\overline{\theta}\tau + \kappa\nabla\overline{\theta}\cdot\nabla\tau + u\cdot\nabla\overline{\theta}\tau &= \int_E \delta(\tau+k)\tau + \kappa\nabla\tau\cdot\nabla\tau + \tilde{\mathbf{u}}\cdot\nabla\tau\tau \\ &= \int_E \delta|\tau|^2 + \kappa|\nabla\tau|^2 + \delta k\tau + \tilde{\mathbf{u}}\cdot\nabla\tau\tau \\ &= \int_\Omega \delta|\tau|^2 + \kappa|\nabla\tau|^2 + \underbrace{\delta k\tau}_{\geq 0} + \underbrace{\int_\Omega \tilde{\mathbf{u}}\cdot\nabla\tau\tau}_{=0} \geq \delta\|\tau\|^2 + \kappa\|\nabla\tau\|^2.\end{aligned}$$

From the assumption follows that either

$$(i) : 0 \leq \delta\|\tau\|^2 \leq \langle f_\tau, \tau\rangle - \kappa\|\nabla\tau\|^2 \leq 0 \;\text{ or }\; (ii) : 0 \leq \kappa\|\nabla\tau\|^2 \leq \langle f_\tau, \tau\rangle - \delta\|\tau\|^2 \leq 0.$$

In both cases, we obtain $\tau = 0$ a.e. on Ω, which is equivalent to $\overline{\theta}^+ \leq k$ a.e. on Ω. However, this is a contradiction to the assertion $k < \text{esssup}_\Omega(\overline{\theta}) \leq \text{esssup}_\Omega(\overline{\theta}^+)$. Together with step (i) we obtain

$$\text{esssup}_\Omega(\overline{\theta}) \leq \sup_{\Gamma_D}(\overline{\theta}^+) \leq \sup_{\Gamma_D}(\theta_b^+).$$

Step (iii) : $\text{essinf}_\Omega\overline{\theta} \geq \inf_{\Gamma_D}(\theta_b^-)$.
Setting $\tilde{f}_\tau := -f_\tau$, $\tilde{\theta} := -\overline{\theta}$, $\tilde{\theta}_b := -\theta_b$, multiplying (2.3) by -1 and applying the previous steps yields

$$\text{esssup}_\Omega(\tilde{\theta}) \leq \sup_{\Gamma_D}(\tilde{\theta}_b^+).$$

Using $\text{esssup}_\Omega(\tilde{\theta}) = -\text{essinf}_\Omega(\overline{\theta})$, $\tilde{\theta}_b^+ = -\theta_b^-$ and $\inf_{\Gamma_D}(u) = -\sup_{\Gamma_D}(-u)$, we obtain

$$-\text{essinf}_\Omega\overline{\theta} \leq -\inf_{\Gamma_D}(\theta_b^-).$$

□

2.4 Existence of Solutions

Proving existence of solutions to (2.2) can be achieved by application of the well-known Galerkin principle, following the work in [19]. Here, the application of the general fixed-point theorem 2.16 to a finite dimensional version of Problem 2.5 is combined with an approximation of the infinite dimensional problem by a series of finite dimensional systems.

Theorem 2.16. *(Leray-Schauder fixed-point theorem, Theorem 6.16 in [14])*
Let Y be a Hilbert space and let $\mathcal{F}\colon Y \to Y$ be a compact map. Consider the fixed-point problem: find $y^* \in Y$ such that

$$y^* = \mathcal{F}(y^*) \tag{2.4}$$

Associate with (2.4) the family of fixed-point problems: find $y_\lambda \in Y$ such that

$$y_\lambda = \lambda \mathcal{F}(y_\lambda), 0 \leq \lambda \leq 1. \tag{2.5}$$

If there is a constant K such that all solutions of (2.5) are uniformly bounded, i.e. $\|y_\lambda\| \leq K$ for all $0 \leq \lambda \leq 1$, then there exists a solution to (2.4).

Proposition 2.17. *(Existence of solutions in finite dimensional spaces)*
Let $\mathbf{W} \subset \mathbf{V}$, $T \subset \Theta$ denote finite dimensional Hilbert spaces. Let Assumptions 2.9 and 2.10 hold and assume that the boundary lifting θ_b is chosen such that $\|\theta_b\|_3 < d(a_{\mathbf{F}})$ for d defined in Proposition 2.11. Then, there exists a solution (u, θ) for Problem 2.5 with $\lambda = 1$, i.e.

$$(\mathbf{u}, \theta) = \mathcal{F}[\mathbf{W}, T](u, \theta) \tag{2.6}$$

Proof. Follows by application of Proposition 2.11 with Remark 2.12 (uniform stability), Lemma 2.8 (compactness of $\mathcal{F}[\mathbf{W}, T]\colon Y \to Y$ with $Y := \mathbf{W} \times T$) and Theorem 2.16. □

The following Theorem is based on the existence result in [19], with a slight modification to take into account the more general body force term $\mathbf{F}$.

Theorem 2.18. *(Existence of solutions for stationary Boussinesq equations)*
Let Assumptions 2.9 and 2.10 hold and assume that the boundary lifting θ_b is chosen such that $\|\theta_b\|_3 < d(a_{\mathbf{F}})$ for d defined in Proposition 2.11. Then, there exists a solution $(\mathbf{u}, \theta) \in \mathbf{V} \times \Theta$ for Problem 2.2 with $\lambda = 1$.

Proof. We only show the proof for $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = u$. The other cases follow analogously.

Since $\mathbf{V}$ and Θ are closed subspaces of the separable normed spaces $W^{1,2}(\Omega)^d$ and $W^{1,2}(\Omega)$, respectively, they are separable as well according to Lemma A.1. Therefore, there a sequences $(\mathbf{W}_m)_m, (T_m)_m$ of finite dimensional subspaces satisfying

$$\mathbf{W}_m \subset \mathbf{V},\ \mathbf{W}_m \subset \mathbf{W}_{m+1} \text{ and } T_m \subset \Theta,\ T_m \subset T_{m+1}$$

with

$$\mathbf{V} = \overline{\bigcup_{n\in\mathbb{N}} \mathbf{W}_n} \text{ and } \Theta = \overline{\bigcup_{n\in\mathbb{N}} T_n}.$$

For each $n \in \mathbb{N}$ let $(\mathbf{u}_n, \theta_n) \in \mathbf{W}_n \times T_n$ denote the solution of Problem 2.2 with $\mathbf{V} \times \Theta$ replaced by $\mathbf{W}_n \times T_n$. These solutions exist due to Proposition 2.17. Moreover, they are uniformly bounded according to Proposition 2.11 and Remark 2.12,

$$\|\nabla \mathbf{u}_n\| + \|\nabla \theta_n\| \leq G_{\mathbf{u}} + G_\theta \text{ for all } n \in \mathbb{N}.$$

Since $\mathbf{V}$ and Θ are Hilbert spaces, they are reflexive. Thus, there are $(\mathbf{u}_*, \theta_*) \in \mathbf{V} \times \Theta$ and a subsequence $(\mathbf{u}_k, \theta_k)_k \subset (\mathbf{u}_n, \theta_n)_n$ with $\mathbf{u}_k \rightharpoonup \mathbf{u}_*$ in $\mathbf{V}$ and $\theta_k \rightharpoonup \theta_*$ in Θ. Due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ for $d \in \{2, 3\}$, we additionally have $\mathbf{u}_k \to \mathbf{u}_*$ in $L^4(\Omega)^d$ and $\theta_k \to \theta_*$ in $L^4(\Omega)$. Since $(\mathbf{u}_k, \theta_k)$ are solutions for the finite dimensional problems, there holds

$$a_{\mathbf{v}}(\mathbf{u}_k, \mathbf{v}) + c_{\mathbf{v}}(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) - \langle \mathbf{F}(\theta_k + \theta_b) + \mathbf{f}_{\mathbf{v}}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathbf{W}_n,\ n \leq k$$
$$a_\tau(\theta_k + \theta_b, \tau) + c_\tau(\mathbf{u}_k, \theta_k + \theta_b, \tau) - \langle f_\tau, \tau \rangle = 0 \quad \forall \tau \in \theta_n,\ n \leq k$$

where we used the equivalence between variational formulation 2.2 and fixed-point formulation 2.5 due to Lemma 2.8. Let $n \in \mathbb{N}$ and $(\mathbf{v}, \tau) \in \mathbf{W}_n \times T_n$ be arbitrary. Due to weak convergence, we obtain for $k \to \infty$ convergence of the linear terms according to $(\mathbf{u}_k, \mathbf{v}) \to (\mathbf{u}_*, \mathbf{v})$, $(\theta_k, \tau) \to (\theta_*, \tau)$, $a_v(\mathbf{u}_k, \mathbf{v}) \to a_v(\mathbf{u}_*, \mathbf{v})$, $a_\tau(\theta_k, \mathbf{v}) \to a_\tau(\theta_*, \tau)$ and $c_\tau(\mathbf{u}_k, \theta_b, \tau) \to c_\tau(\mathbf{u}_*, \theta_b, \tau)$.

By Assumption 2.9 (iv) we have $\langle \mathbf{F}(\theta_k + \theta_b), \mathbf{v}\rangle_{\mathbf{U}^*} \to \langle \mathbf{F}(\theta_*), \mathbf{v}\rangle_{\mathbf{U}^*}$. Finally,

$$\begin{aligned} |c_{\mathbf{v}}(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) - c_{\mathbf{v}}(\mathbf{u}_*, \mathbf{u}_*, \mathbf{v})| &\le |c_{\mathbf{v}}(\mathbf{u}_k - \mathbf{u}_*, \mathbf{u}_k, \mathbf{v})| + |c_{\mathbf{v}}(\mathbf{u}_*, \mathbf{u}_k - \mathbf{u}_*, \mathbf{v})| \\ &\le \|\mathbf{u}_k - \mathbf{u}_*\|_4 \|\nabla \mathbf{u}_k\| \|\mathbf{v}\|_4 + |c_{\mathbf{v}}(\mathbf{u}_*, \mathbf{v}, \mathbf{u}_k - \mathbf{u}_*)| \\ &\le \|\mathbf{u}_k - \mathbf{u}_*\|_4 \, G_{\mathbf{u}} \, \|\mathbf{v}\|_4 + \|\mathbf{u}_*\|_4 \|\nabla \mathbf{v}\| \|\mathbf{u}_k - \mathbf{u}_*\|_4 \\ &\to 0 \end{aligned}$$

and, similarly, $c_\tau(\mathbf{u}_k, \theta_k, \tau) \to c_\tau(\mathbf{u}_*, \theta_*, \tau)$. Summing up,

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{u}_*, \mathbf{v}) + c_{\mathbf{v}}(\mathbf{u}_*, \mathbf{u}_*, \mathbf{v}) - \langle \mathbf{F}(\theta_* + \theta_b) + \mathbf{f}_{\mathbf{v}}, \mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in \mathbf{W}_n,\ \forall n \in \mathbb{N} \\ a_\tau(\theta_* + \theta_b, \tau) + c_\tau(\mathbf{u}_*, \theta_* + \theta_b, \tau) - \langle f_\tau, \tau\rangle = 0 \quad \forall \tau \in \theta_n,\ \forall n \in \mathbb{N} \end{aligned} \tag{2.7}$$

Since $\bigcup_n \mathbf{W}_n \times \bigcup_n T_n$ is dense in $\mathbf{V} \times \Theta$, (2.7) holds for all $(\mathbf{v}, \tau) \in \mathbf{V} \times \Theta$. □

3 The TEHD Boussinesq Problem

In this section, we consider the combined problem of finding a solution $(\mathbf{u}, \theta, \Phi)$ of the stationary TEHD equations. Existence of solutions is shown by applying a fixed-point iteration that is alternating between solutions $(\mathbf{u}, \theta)$ of the Boussinesq problem 2.2 and solutions Φ of Gauss' law. Afterward, we show that solutions are unique under certain restrictions onto the data. As final step in this section, we propose certain ways of fitting the DEP force into the general force term $\mathbf{F}$.

3.1 Variational Formulation and Existence of Solutions

In addition to spaces $\mathbf{U}, \mathbf{V}, M, \Theta$ used in the previous sections, we define the potential space $\Upsilon := H^1_D(\Omega)$ and for given $\epsilon \in C(\mathbb{R}, \mathbb{R})$ the mapping

$$a_\beta \colon L^\infty(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \quad (\theta, \Phi, \beta) \mapsto (\epsilon(\theta)\nabla\Phi, \nabla\beta).$$

Problem 3.1. *(Stationary TEHD equations)*
Let $\theta_b \in C^1(\overline{\Omega})$ and $\Phi_b \in C^1(\overline{\Omega})$ denote liftings of given boundary conditions and $\mathbf{F} \colon \Theta \times \Upsilon \to \mathbf{U}^*, \mathbf{f}_{\mathbf{v}} \in \mathbf{U}^*, f_\tau \in \Theta^*, f_\beta \colon L^2(\Omega) \to \Upsilon^*$ be given body forces and $\epsilon \in C(\mathbb{R}, \mathbb{R})$. Let either $\bar{\mathbf{u}}, \tilde{\mathbf{u}} \in \mathbf{V}, \bar{\theta} \in \Theta, \bar{\Phi} \in \Upsilon$ denote fixed functions or the unknown variables $\mathbf{u}, \theta, \Phi$.

For $\delta \ge 0$, $\nu, \kappa, \gamma > 0$ find $\mathbf{u} \in \mathbf{V}$, $\theta \in \Theta \cap L^\infty(\Omega), \Phi \in \Upsilon$ such that for all $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_v(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) &= \langle \mathbf{F}(\theta + \theta_b, \bar{\Phi} + \Phi_b) + \mathbf{f}_{\mathbf{v}}, \mathbf{v}\rangle_{\mathbf{U}^*} \\ a_\tau(\theta + \theta_b, \tau) + c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_\tau, \tau\rangle_{\Theta^*} \\ a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_\beta(\theta + \theta_b), \beta\rangle_{\Upsilon^*} \end{aligned}$$

Assumption 3.2. *(Permittivity ϵ)*
Let $\epsilon \colon \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with constant L_ϵ.

Assumption 3.3. *(Source term f_τ)*
For the temperature source term $f_\tau \in \Theta^*$ we define conditions

(i) $\delta > 0$ *and* $|\langle f_\tau, \tau\rangle_{\Theta^*}| \le \kappa \|\nabla\tau\|^2$ *for all* $\tau \in \Theta$

(ii) $|\langle f_\tau, \tau\rangle_{\Theta^*}| \le \delta \|\tau\|^2$ *for all* $\tau \in \Theta$

Assumption 3.4. *(Source term f_β)*
Let the potential source term $f_\beta \colon L^2(\Omega) \to \Upsilon^*$ *satisfy*

(i) There are constant $a_f, b_f \ge 0$ *such that for all* $\theta \in L^2(\Omega)$ *and* $\beta \in \Upsilon$

$$|\langle f_\beta(\theta), \beta\rangle_{\Upsilon^*}| \le (a_f\|\theta\| + b_f)\|\nabla\beta\|.$$

(ii) Let a sequence $(\theta_n)_n \subset H^1(\Omega)$ *be given that converges to* $\theta_* \in H^1(\Omega)$ *in the following sense*

$$\theta_n \rightharpoonup \theta_* \text{ in } H^1,\ \theta_n \to \theta_* \text{ in } L^4 \text{ and } \|\theta_n\|_{1,2} \le K \text{ for all } n \in \mathbb{N}.$$

Then, for all $\beta \in \Upsilon$ *there holds true*

$$|\langle f_\beta(\theta_*) - f_\beta(\theta_n), \beta\rangle_{\Upsilon^*}| \to 0.$$

(iii) f_β *is Lipschitz continuous in the following sense: for all* $D > 0$ *there is* $L_\beta \ge 0$*, such that*

$$|\langle f_\beta(\theta_1) - f_\beta(\theta_2), \beta\rangle_{\Upsilon^*}| \le L_\beta\|\theta_1 - \theta_2\|_{1,2}\|\nabla\beta\| \text{ for all } \theta_1, \theta_2 \in B_D(0, H^1(\Omega)) \text{ and } \beta \in \Upsilon.$$

Theorem 3.5. *(Existence and stability of stationary TEHD solutions)*
Let the assertions in Problem 3.1 and Assumptions 2.9, 2.10, 3.2 and 3.4 hold. Let $\theta_- := \inf_{\Gamma_D}(\theta_b^-)$*,* $\theta_+ := \sup_{\Gamma_D}(\theta_b^+)$ *and* $\theta_\infty := \max\{\theta_+, |\theta_-|\}$*. Define* $\epsilon_- \le \epsilon_+$ *such that*

$$\epsilon([\theta_-, \theta_+]) \subset [\epsilon_-, \epsilon_+].$$

Assume that θ_b *is chosen such that* $\epsilon_- > 0$ *and* $\|\theta_b\|_3 < d(a_{\mathbf{F}}(\tilde G_\Phi))$ *holds with*

$$\begin{aligned}\tilde G_\Phi &= \sqrt{K_2^2 + 1}\, G_\Phi + \|\Phi_b\|_{1,2}\\ G_\Phi &= \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}(a_f|\Omega|^{\frac12}\theta_\infty + b_f).\end{aligned}$$

Moreover, if $\bar\theta = \theta$ *let either Assumption 3.3 (i) or (ii) hold. If* $\bar\theta \in \Theta$ *is fixed, assume that* $\bar\theta \in L^\infty(\Omega)$ *with* $(\bar\theta + \theta_b)(x) \in [\theta_-, \theta_+]$ *a.e.*

Then, there exists a solution $(\mathbf{u}, \theta, \Phi)$ *of Problem 3.1. Further, all solutions of 3.1 satisfy*

$$\begin{aligned}\|\nabla\mathbf{u}\| &\le g_{\mathbf{u}}(\|\theta_b\|_3, a_{\mathbf{F}}(\tilde G_\Phi), b_{\mathbf{F}}(\tilde G_\Phi), \|\theta_b\|_{1,2}, \|f_\tau\|_{\Theta^*}, \|\mathbf{f_v}\|_{\mathbf{U}^*}) = G_{\mathbf{u}},\\ \|\nabla\theta\| &\le g_\theta(\|\theta_b\|_3, a_{\mathbf{F}}(\tilde G_\Phi), b_{\mathbf{F}}(\tilde G_\Phi), \|\theta_b\|_{1,2}, \|f_\tau\|_{\Theta^*}, \|\mathbf{f_v}\|_{\mathbf{U}^*}) = G_\theta,\\ \|\nabla\Phi\| &\le G_\Phi,\\ (\theta + \theta_b)(x) &\in [\theta_-, \theta_+] \text{ a.e.}\end{aligned}$$

with functions $g_{\mathbf{u}}, g_\theta$ *defined in Proposition 2.11.*

Proof. As before, we only show the proof for the implicit case $\bar{\mathbf{u}} = \tilde{\mathbf{u}} = u, \bar\Phi = \Phi$.

We split Problem 3.1 into two parts: for given $\Phi_\# \in \Upsilon$ find $(\mathbf{u}, \theta) \in \mathbf{V} \times \Theta$ satisfying

$$(P_1): \begin{cases} a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_v(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \langle \mathbf{F}(\theta + \theta_b, \Phi_\# + \Phi_b) + \mathbf{f_v}, \mathbf{v}\rangle_{\mathbf{U}^*} &= 0 \quad \forall \mathbf{v} \in \mathbf{V}\\ a_\tau(\theta + \theta_b, \tau) + c_\tau(\mathbf{u}, \theta + \theta_b, \tau) - \langle f_\tau, \tau\rangle_{\Theta^*} &= 0 \quad \forall \tau \in \Theta\end{cases}$$

and for given $(\bar{\theta}_\#, \theta_\#) \in \Theta \times \Theta$ find $\Phi \in \Upsilon$ such that

$$(P_2) : a_\beta(\bar{\theta}_\# + \theta_b, \Phi + \Phi_b, \beta) = f_\beta(\theta_\# + \theta_b) \quad \forall \beta \in \Upsilon$$

We define the following fixed-point iteration: let $\Phi_0 \in \Upsilon$ be arbitrary and set for $n \in \mathbb{N}$

- $(\mathbf{u}_n, \theta_n)$ denotes the solution of (P_1) for $\Phi_\# = \Phi_{n-1}$
- if $\bar{\theta} = \theta$, then $\bar{\theta}_n := \theta_n$. Otherwise, $\bar{\theta}_n := \bar{\theta}$
- Φ_n denotes the solution of (P_2) for $(\bar{\theta}_\#, \theta_\#) = (\bar{\theta}_n, \theta_n)$

Here, Theorem 2.18 garuantees the existence of $(\mathbf{u}_n, \theta_n)$. Moreover, according to the weak maximum principle, Proposition 2.15, in combination with the assumptions on θ_b and f_τ for the case $\bar{\theta} = \theta$, or the assumptions on $\bar{\theta}$ if this variable is fixed, we have

$$(\bar{\theta} + \theta_b)(x) \in [\theta_-, \theta_+] \text{ a.e. and } (\bar{\theta}_n + \theta_b)(x) \in [\theta_-, \theta_+] \text{ a.e. for all } n \in \mathbb{N}.$$

Therefore, $\epsilon_n := \epsilon(\bar{\theta}_n + \theta_b) \in L^\infty(\Omega)$ and $\epsilon_n \geq \epsilon_- > 0$ a.e.

For such kind of $\bar{\theta}_n$, the bilinear form $\Upsilon \times \Upsilon \ni (\Phi, \beta) \mapsto a_\beta(\bar{\theta}_n + \theta_b, \Phi, \beta) \in \mathbb{R}$ is bounded and coercive. Thus, there exists a unique solution $\Phi_n \in \Upsilon$ of (P_2) by Lax-Milgram and it is bounded according to

$$\|\nabla \Phi_n\| \leq \frac{\epsilon_+}{\epsilon_-} \|\nabla \Phi_b\| + \frac{1}{\epsilon_-}(a_f \|\theta_n + \theta_b\| + b_f) \leq G_\Phi.$$

On the other hand, we have for all $n \in \mathbb{N}$

$$\|\nabla \mathbf{u}_n\| \leq G_\mathbf{u} \text{ and } \|\nabla \theta_n\| \leq G_\theta,$$

according to Proposition 2.11 by using $\|\Phi_n + \Phi_b\|_{1,2} \leq \tilde{G}_\Phi$ and the monotonicity of $a_\mathbf{F}, b_\mathbf{F}, g_\mathbf{u}, g_\theta$. As in the proof of Theorem 2.18, there are $(\mathbf{u}_*, \theta_*, \Phi_*) \in \mathbf{V} \times \Theta \times \Upsilon$ and a subsequence $(\mathbf{u}_k, \theta_k, \Phi_k)_k \subset (\mathbf{u}_n, \theta_n, \Phi_n)_n$ with $\mathbf{u}_k \rightharpoonup \mathbf{u}_*$ in $\mathbf{V}$, $\theta_k \rightharpoonup \theta_*$ in Θ and $\Phi_k \rightharpoonup \Phi_*$ in Υ. Due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ for $d \in \{2, 3\}$, we additionally have $\mathbf{u}_k \to \mathbf{u}_*$ in $L^4(\Omega)^d$, $\theta_k \to \theta_*$ in $L^4(\Omega)$ and $\Phi_k \to \theta_*$ in $L^4(\Omega)$. By Assumption 2.9 (iii),

$$|\langle \mathbf{F}(\theta_* + \theta_b, \Phi_* + \Phi_b) - \mathbf{F}(\theta_n + \theta_b, \Phi_n + \Phi_b), \mathbf{v} \rangle_{\mathbf{U}^*}| \to 0 \text{ for all } \mathbf{v} \in \mathbf{V}.$$

Thus, as in the proof of Theorem 2.18, $(\mathbf{u}_*, \theta_*)$ solves (P_1) for $\Phi_\# = \Phi_*$ and it holds

$$\epsilon((\theta_* + \theta_b)(x)) \in [\epsilon_-, \epsilon_+] \text{ a.e.}$$

since Proposition 2.15 also applies for θ_*.

It remains to show that Φ_* solves (P_2) for $(\bar{\theta}_\#, \theta_\#) = (\bar{\theta}_*, \theta_*)$ with $\bar{\theta}_* = \theta_*$ if $\bar{\theta} = \theta$ or $\bar{\theta}_* = \bar{\theta}$ otherwise. In the former case, let $\beta \in C_D^{0,1}(\overline{\Omega}) \cap W^{1,6}(\Omega)$ be arbitrary but fixed. Then, using Hölder's inequaility and the Lipschitz continuity of ϵ,

$$\begin{aligned}
&|a_\beta(\theta_k + \theta_b, \Phi_k + \Phi_b, \beta) - a_\beta(\theta_* + \theta_b, \Phi_* + \Phi_b, \beta)| \\
\leq &|((\epsilon(\theta_k + \theta_b) - \epsilon(\theta_* + \theta_b))\nabla(\Phi_k + \Phi_b), \nabla\beta)| + |(\epsilon(\theta_* + \theta_b)\nabla(\Phi_k - \Phi_*), \nabla\beta)| \\
\leq &\|\epsilon(\theta_k + \theta_b) - \epsilon(\theta_* + \theta_b)\|_3 \|\nabla(\Phi_k + \Phi_b)\| \|\nabla\beta\|_6 + |(\epsilon(\theta_* + \theta_b)\nabla(\Phi_k - \Phi_*), \nabla\beta)| \\
\leq &L_\epsilon G_\Phi \|\theta_k - \theta_*\|_3 \|\nabla\beta\|_6 + |(\epsilon(\theta_* + \theta_b)\nabla(\Phi_k - \Phi_*), \nabla\beta)| \\
\to &0
\end{aligned}$$

Here, convergence of both terms is implied by $\theta_k \to \theta_b$ in $L^4(\Omega)$ and by $\Phi_k \rightharpoonup \Phi_*$ in Υ, respectively.

By the assertion on f_β, we additionally have

$$\langle f_\beta(\theta_k + \theta_b), \beta\rangle_{\Upsilon^*} \to \langle f_\beta(\theta_* + \theta_b), \beta\rangle_{\Upsilon^*}.$$

Combining both results yields

$$a_\beta(\theta_* + \theta_b, \Phi_* + \Phi_b, \beta) = \langle f_\beta(\theta_* + \theta_b), \beta\rangle_{\Upsilon^*} \text{ for all } \beta \in C_D^{0,1}(\overline{\Omega}) \cap W^{1,6}(\Omega).$$

Since $C_D^{0,1}(\overline{\Omega}) \cap W^{1,6}(\Omega)$ is dense in $H_D^1(\Omega)$ and the linear form $H_D^1(\Omega) \ni \beta \mapsto a_\beta(\theta_* + \theta_b, \Phi_* + \Phi_b, \beta)$ is continuous, Φ_* solves (P_2) for $\theta_\# = \theta_*, \overline{\theta}_\# = \overline{\theta}_*$.

In order to show the stated energy norm estimate, let $(\mathbf{u}, \theta, \Phi) \in \mathbf{V} \times \Theta \times \Upsilon$ denote an arbitrary solution. According to Proposition 2.15, $\|\theta + \theta_b\|_\infty \le \theta_\infty$. Thus, as for Φ_n, we have that

$$\|\nabla\Phi\| \le \frac{\epsilon_+}{\epsilon_-}\|\nabla\Phi_b\| + \frac{1}{\epsilon_-}(a_f\|\theta + \theta_b\| + b_f) \le G_\Phi,$$

which implies $\|\Phi + \Phi_b\|_{1,2} \le \tilde{G}_\Phi$.

Moreover, by means of Proposition 2.11,

$$\begin{aligned}\|\nabla\mathbf{u}\| &\le g_{\mathbf{u}}(\|\theta_b\|_3, a_{\mathbf{F}}, b_{\mathbf{F}}, \|\theta_b\|_{1,2}, \|f_\tau\|_{\Theta^*}, \|\mathbf{f_v}\|_{\mathbf{U}^*})\\ \|\nabla\theta\| &\le g_\theta(\|\theta_b\|_3, a_{\mathbf{F}}, b_{\mathbf{F}}, \|\theta_b\|_{1,2}, \|f_\tau\|_{\Theta^*}, \|\mathbf{f_v}\|_{\mathbf{U}^*})\end{aligned}$$

with constants $a_{\mathbf{F}} = a_{\mathbf{F}}(\|\Phi + \Phi_b\|_{1,2}) \le a_{\mathbf{F}}(\tilde{G}_\Phi)$ and $b_{\mathbf{F}} = b_{\mathbf{F}}(\|\Phi + \Phi_b\|_{1,2}) \le b_{\mathbf{F}}(\tilde{G}_\Phi)$. Since $g_{\mathbf{u}}, g_\theta$ are non-decreasing in their arguments x_2 and x_3, we obtain the stated result. □

By means of a standard procedure, see e.g. Lemma IX.1.2 in [10], existence of solutions for the problem in mixed form can be shown.

Proposition 3.6. *(Recovering the pressure)*
Let $(\mathbf{u}, \theta, \Phi) \in \mathbf{V} \times \Theta \times \Upsilon$ denote a solution of Problem 3.1. Then, there exists a pressure $p \in M$ such that $(\mathbf{u}, p, \theta, \Phi)$ is a solution to the mixed problem

$$\begin{aligned}a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_v(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) &= \langle \mathbf{F}(\theta + \theta_b, \bar{\Phi} + \Phi_b) + \mathbf{f_v}, \mathbf{v}\rangle_{\mathbf{U}^*}\\ a_\tau(\theta + \theta_b, \tau) + c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_\tau, \tau\rangle_{\Theta^*}\\ a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_\beta(\theta + \theta_b), \beta\rangle_{\Upsilon^*}\\ b(\mathbf{u}, q) &= 0\end{aligned}$$

for all $(\mathbf{v}, q, \tau, \beta) \in \mathbf{U} \times M \times \Theta \times \Upsilon$.

Proof. Define the linear operator $B^*\colon M \to \mathbf{U}^*$ via $\langle B^*p, \mathbf{v}\rangle_{\mathbf{U}^*} := b(\mathbf{v}, p) = \frac{1}{\rho_r}(\nabla\cdot\mathbf{v}, p)$ and let a functional $l \in \mathbf{U}^*$ be given as

$$l(\mathbf{v}) := a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_{\mathbf{v}}(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) - \langle \mathbf{F}(\theta + \theta_b, \bar{\Phi} + \Phi_b) + f_\tau, \mathbf{v}\rangle_{\mathbf{U}^*}.$$

Then, $l(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{V}$, i.e. $l \in \mathbf{V}^\circ = \{g^* \in \mathbf{U}^* :\ \langle g, v\rangle_{\mathbf{U}} = 0\ \forall v \in \mathbf{V}\}$. Since the inf-sup condition holds for the space $\mathbf{U} \times M$, B^* is an isomorphism from M to $\mathbf{V}^\circ$ according to Theorem A.4. Thus, there is $p \in M$ such that $B^*p = l$, i.e. $b(\mathbf{v}, p) = l(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{U}$. Moreover, $b(\mathbf{u}, q) = 0$ for all $q \in M$, since $\mathbf{v} \in \mathbf{V}$. □

3.2 Uniqueness of Solutions

We investigate uniqueness of solutions of Problem 3.1. To this end, let the assumptions in Theorem 3.5 hold and let $(\mathbf{u}_1, \theta_1, \Phi_1)$ and $(\mathbf{u}_2, \theta_2, \Phi_2)$ denote two solutions. Using the properties of $\mathbf{F}$ given by Assumption 2.9, there is $L_{\mathbf{F}}^{(\Phi)} > 0$ such that

$$D_{\mathbf{F}} := \sup_{w\in\mathbf{V}} \sup_{\theta\in\Theta} \frac{|\langle \mathbf{F}(\theta+\theta_b, \Phi_1+\Phi_b) - \mathbf{F}(\theta+\theta_b, \Phi_2+\Phi_b), w\rangle_{\mathbf{U}^*}|}{\|\nabla w\| \|\theta+\theta_b\|_{1,2}} \le L_{\mathbf{F}}^{(\Phi)} \|\Phi_1 - \Phi_2\|_{1,2}. \tag{3.1}$$

Applying Lemma 2.13 with $\mathbf{F}_i = \mathbf{F}(\cdot, \Phi_i + \Phi_b)$ and introducing $d_x := x_1 - x_2$ with $x \in \{\mathbf{u}, \theta, \Phi, \bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}\}$ yields

$$\begin{aligned} \|\nabla d_{\mathbf{u}}\| &\le D_1 \|\nabla d_{\bar{\mathbf{u}}}\| + D_2 \|\nabla d_{\tilde{\mathbf{u}}}\| + D_5 \|\nabla d_{\bar{\Phi}}\| \\ \|\nabla d_\theta\| &\le D_4 \|\nabla d_{\tilde{\mathbf{u}}}\|. \end{aligned} \tag{3.2}$$

with $D_i,\ i \in \{1,2,3,4\}$ given by (2.2) and $D_5 := \frac{1}{\nu} L_{\mathbf{F}}^{(\Phi)} \left(G_\theta(K_2^2+1) + \sqrt{K_2^2+1} \|\theta_b\|_{1,2} \right)$. Here, $d_y = 0$ for $y \in \{\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}\}$ if the corresponding variable y is fixed. Otherwise, $d_{\bar{\mathbf{u}}} = d_{\mathbf{u}},\ d_{\tilde{\mathbf{u}}} = d_\theta,\ d_{\bar{\theta}} = d_\theta,\ d_{\bar{\Phi}} = d_\Phi$, respectively.

Coming to Gauss's law, according to the assumptions and results from Theorem 3.5, there holds $0 < \epsilon_- \le \epsilon(\theta_i(x) + \theta_b(x)) \le \epsilon_+$ a.e. Thus,

$$\begin{aligned} \epsilon_- \|\nabla d_\Phi\|^2 &\le (\epsilon(\theta_2+\theta_b)\nabla d_\Phi, \nabla d_\Phi) \\ &= \langle f_\beta(\theta_1+\theta_b) - f_\beta(\theta_2+\theta_b), d_\Phi\rangle - ((\epsilon(\theta_1+\theta_b) - \epsilon(\theta_2+\theta_b))\nabla(\Phi_1+\Phi_b), \nabla d_\Phi) \\ &\le L_\beta \|d_\theta\|_{1,2} \|\nabla d_\Phi\| - ((\epsilon(\theta_1+\theta_b) - \epsilon(\theta_2+\theta_b))\nabla(\Phi_1+\Phi_b), \nabla d_\Phi). \end{aligned}$$

If we assume that $\Phi_1 \in W^{1,3}(\Omega)$, then the previous inequality together with $H^1 \hookrightarrow L^6$ implies

$$\epsilon_- \|\nabla d_\Phi\|^2 \le L_\beta \sqrt{K_2^2+1} \|\nabla d_\theta\| \|\nabla d_\Phi\| + L_\epsilon K_6 \|\nabla d_{\bar{\theta}}\| \|\nabla(\Phi_1+\Phi_b)\|_3 \|\nabla d_\Phi\| \tag{3.3}$$

for constants $\alpha_1, \alpha_2 \ge 0$. If Φ is less regular, we may only deduce

$$\epsilon_- \|\nabla d_\Phi\|^2 \le L_\beta \sqrt{K_2^2+1} \|\nabla d_\theta\| \|\nabla d_\Phi\| + L_\epsilon (G_\Phi + \|\nabla\Phi_b\|) \|d_{\bar{\theta}}\|_\infty \|\nabla d_\Phi\|. \tag{3.4}$$

Letting $\|d_{\bar{\theta}}\|_* \in \{\|\nabla d_{\bar{\theta}}\|, \|d_{\bar{\theta}}\|_\infty\}$, we summarize the previous estimates (3.2), (3.3), (3.4) as

$$\begin{aligned} \|\nabla d_{\mathbf{u}}\| &\le \alpha_1 \|\nabla d_{\bar{\mathbf{u}}}\| + \alpha_2 \|\nabla d_{\tilde{\mathbf{u}}}\| + \alpha_3 \|\nabla d_{\bar{\Phi}}\| \\ \|\nabla d_\theta\| &\le \alpha_4 \|\nabla d_{\tilde{\mathbf{u}}}\| \\ \|\nabla d_\Phi\| &\le \alpha_5 \|\nabla d_\theta\| + \alpha_6(\|\nabla\Phi_1\|_3) \|d_{\bar{\theta}}\|_* \end{aligned} \tag{3.5}$$

with constants given by

$$\begin{aligned} &\alpha_1 = D_1,\ \alpha_2 = D_2,\ \alpha_3 = D_5,\ \alpha_4 = D_4 \\ &\alpha_5 = \frac{L_\beta}{\epsilon_-} \sqrt{K_2^2+1} \\ &\alpha_6(s) = \frac{L_\epsilon}{\epsilon_-} \begin{cases} K_6(s + \|\nabla\theta_b\|_3), & \|d_{\bar{\theta}}\|_* = \|\nabla d_{\bar{\theta}}\| \\ G_\Phi + \|\nabla\Phi_b\|, & \|d_{\bar{\theta}}\|_* = \|d_{\bar{\theta}}\|_\infty \end{cases}. \end{aligned} \tag{3.6}$$

Based on the set of inequalities (3.5), the following theorem yields conditions under which uniqueness of solutions for the stationary TEHD Problem 3.1 holds. Hereby, one needs to differentiate w.r.t. the degree of implicitness in 3.1.

Theorem 3.7. *(Uniqueness for small data)*
Let the assertions of Theorem 3.5 hold and constants α_i be given as in (3.5). Then, solutions of Problem 3.1 are unique under certain conditions:

(i) $\tilde{\mathbf{u}} \in \mathbf{V}$ *fixed*

(i.i) $\bar{\mathbf{u}} \in \mathbf{V}$ *fixed: without restriction*

(i.ii) $\bar{\mathbf{u}} = \mathbf{u}$*:* $\alpha_1 < 1$

(ii) $\tilde{\mathbf{u}} = \mathbf{u}$

(ii.i) $\bar{\theta} \in \Theta$ *fixed:* $\alpha_1 + \alpha_2 + \delta_{\Phi,\bar{\Phi}}\alpha_3\alpha_5\alpha_4 < 1$

(ii.ii) $\bar{\theta} = \theta$

(ii.ii.i) $\bar{\Phi} \in \Upsilon$ *fixed:* $\alpha_1 + \alpha_2 < 1$

(ii.ii.ii) $\bar{\Phi} = \Phi$

(ii.ii.ii.i) $d = 2$*:* $\alpha_1 + \alpha_2 + \alpha_3\alpha_4 (\alpha_5 + \alpha_6 K_\infty) < 1$

(ii.ii.ii.ii) $d = 3$*:* $R > 0$ *such that* $\alpha_1 + \alpha_2 + \alpha_3\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_6(R) < 1$ *and* $\Phi \in B_R(0, W^{1,3}(\Omega))$*.*

Proof. The assertions are proven by using (3.5) with $d_x = 0$ if $x \in \{\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\theta}, \bar{\Phi}\}$ is fixed, or $d_{\bar{x}} = d_x$ for $x \in \{\mathbf{u}, \theta, \Phi\}$ otherwise. In case (i), i.e. $d_{\tilde{\mathbf{u}}} = 0$, (3.5) directly leads to $d_\theta = d_{\bar{\theta}} = d_\Phi = d_{\bar{\Phi}} = 0$. For $(i.i)$, also $d_{\bar{\mathbf{u}}} = 0$, implying $d_{\mathbf{u}} = d_\theta = d_\Phi = 0$ and therefore $(\mathbf{u}_1, \theta_1, \Phi_1) = (\mathbf{u}_2, \theta_2, \Phi_2)$. In case $(i.ii)$, we obtain

$$\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\|,$$

i.e. $d_{\mathbf{u}} = 0$ if $1 > \alpha_1$. In $(ii.i)$, (3.5) leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| + \delta_{\Phi,\bar{\Phi}}\alpha_3\alpha_5\alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq \alpha_5 \|\nabla d_\theta\|
\end{aligned}$$

with $\delta_{x,y}$ denoting the Kronecker delta. Thus, uniqueness of solutions is given if $\alpha_1 + \alpha_2 + \delta_{\Phi,\bar{\Phi}}\alpha_3\alpha_5\alpha_4 < 1$. If $\bar{\theta} = \theta$ and $d_{\bar{\Phi}} = 0$, i.e. case $(ii.ii.i)$, (3.5) with $\|d_\theta\|_* = \|d_\theta\|_\infty$ leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq (\alpha_5 + \alpha_6(\Phi_1)) \|d_\theta\|_\infty.
\end{aligned}$$

Now, $\alpha_1 + \alpha_2 < 1$ implies $d_{\mathbf{u}} = 0$ and consequently, $d_\theta = 0$ and $d_\Phi = 0$. Finally, consider the case $(ii.ii.ii)$, i.e. the fully implicit problem. If $d = 2$, then $\|d_\theta\|_\infty \leq M_\infty \|\nabla d_\theta\|$ by the Sobolev embedding $H^1(\Omega) \hookrightarrow C^0(\Omega)$. Therefore, (3.5) with $\|d_\theta\|_* = \|d_\theta\|_\infty$ leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| + \alpha_3\alpha_4 (\alpha_5 + \alpha_6 M_\infty) \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq (\alpha_5 + \alpha_6 M_\infty) \|\nabla d_\theta\|.
\end{aligned}$$

Thus, under the condition $\alpha_1 + \alpha_2 + \alpha_3\alpha_4 (\alpha_5 + \alpha_6 M_\infty) < 1$, there holds $d_{\mathbf{u}} = d_\theta = d_\Phi = 0$. If $d = 3$ and $\Phi \in W^{1,3}(\Omega)$, (3.5) with $\|d_\theta\|_* = \|\nabla d_\theta\|$ leads to

$$\begin{aligned}
&\|\nabla d_{\mathbf{u}}\| \leq \alpha_1 \|\nabla d_{\mathbf{u}}\| + \alpha_2 \|\nabla d_{\mathbf{u}}\| + \alpha_3\alpha_4 (\alpha_5 + \alpha_6(\|\nabla \Phi_1\|_3) \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\theta\| \leq \alpha_4 \|\nabla d_{\mathbf{u}}\| \\
&\|\nabla d_\Phi\| \leq (\alpha_5 + \alpha_6(\|\nabla \Phi_1\|_3)) \|\nabla d_\theta\|.
\end{aligned}$$

In this case, $d_{\mathbf{u}} = 0$ if $\alpha_1 + \alpha_2 + \alpha_3\alpha_4\alpha_5 < 1$ and $\|\nabla \Phi_1\|_3 < R$ such that

$$\alpha_1 + \alpha_2 + \alpha_3\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_6(R) < 1.$$

□

The previous theorem shows that solutions are unique without any restriction onto the problem data only, if the convection terms in both momentum and heat equation are explicitly given. Otherwise, uniqueness only holds under certain restrictions onto the data. A closer look on the involved constants α_i reveals that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ tend to zero as $\|\theta_b\|_{1,2}$ tends to zero. Thus, if the energy that is put into the system by means of the boundary temperature is sufficiently small, uniqueness of the solution is ensured.

3.3 Modeling of DEP Force

In this section, we propose several approximations to the body force

$$\mathbf{f} = \mathbf{f}_E + \mathbf{f}_G = \alpha_e (\nabla\Phi)^2 \nabla\theta - \alpha_g \mathbf{g}\theta$$

that satisfy Assumption 2.9. By means of Theorem 3.5, we may state existence of solutions $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, θ, $\Phi \in H^1(\Omega)$. For such functions, the term $(\nabla\Phi)^2\nabla\theta$ is not contained in $\mathbf{H}^{-1}(\Omega)$, making it necessary to replace it by an expression $\mathbf{F}(\theta, \Phi)$ that requires less regularity of θ and Φ to be an element of $\mathbf{H}^{-1}(\Omega)$. In order to do so, we make use of an alternative form $\mathbf{f}_{E,a}$ of $\mathbf{f}_E$ which is obtained by subtracting

$$\nabla\left(\alpha_e(\nabla\Phi)^2\theta\right) = 2\alpha_e(\nabla^2\Phi\nabla\Phi)\theta + \alpha_e\left(\nabla\Phi\right)^2\nabla\theta =: -\mathbf{f}_{E,a} + \mathbf{f}_E$$

from the momentum equation and replacing p by the generalized pressure $P := \frac{1}{\rho_r}p - \alpha_e(\nabla\Phi)^2\theta$.

The proposed approximations $\mathbf{F}_i$ rely on the idea of either replacing the potential Φ in $\mathbf{f}_{E,a}$ by some smooth, fixed function Φ_0, or on applying a smoothing operator to Φ. The following definitions make use of a fixed potential Φ_0.

Definition 3.8. *(Standard DEP force with fixed potential)*
For $\Phi_0 \in W^{1,6}(\Omega)$ define

$$\begin{aligned}\mathbf{F}_{s,0}\colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^*\\ (\theta, \Phi) &\mapsto \alpha_e((\nabla\Phi_0)^2\nabla\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

Definition 3.9. *(Alternative DEP force with fixed potential)*
For $\Phi_0 \in W^{2,3}(\Omega)$ define

$$\begin{aligned}\mathbf{F}_{a,0}\colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^*\\ (\theta, \Phi) &\mapsto -2\alpha_e((\nabla^2\Phi_0\nabla\Phi_0)\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

Definition 3.10. *(Linearized alternative DEP force)*
For $\Phi_0 \in W^{2,12}(\Omega)$ define

$$\begin{aligned}\mathbf{F}_{a,1}\colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^*\\ (\theta, \Phi) &\mapsto -2\alpha_e((\nabla^2\Phi_0\nabla\Phi)\theta, \cdot) - \alpha_g(\theta\mathbf{g}, \cdot).\end{aligned}$$

Proposition 3.11. *(Properties of the body force)*
$\mathbf{F}_{s,0}$, $\mathbf{F}_{a,0}$ and $\mathbf{F}_{a,1}$ satisfy Assumption 2.9.

Proof. Assumption *(i)* and *(ii)* directly follow by the linearity of $\mathbf{F}_{s,0}, \mathbf{F}_{a,1}, \mathbf{F}_{a,1}$ w.r.t. θ and their grwoth rate w.r.t. $\|\Phi\|_{1,2}$ being linearly at most. Validness of *(iii)* is shown for $\mathbf{F}_{a,1}$ only: Let sequences $(\theta_n)_n \subset H^1(\Omega)$ and $(\Phi_n)_n \subset H^1(\Omega)$ be given that converge to $\theta_* \in H^1(\Omega)$ and $\Phi_* \in H^1(\Omega)$, respectively, in the following sense

$$\theta_n \rightharpoonup \theta_* \text{ in } H^1,\ \theta_n \to \theta_* \text{ in } L^4 \text{ and } \|\theta_n\|_{1,2} \le K \text{ for all } n \in \mathbb{N}$$
$$\Phi_n \rightharpoonup \Phi_* \text{ in } H^1, \Phi_n \to \Phi_* \text{ in } L^4 \text{ and } \|\Phi_n\|_{1,2} \le K \text{ for all } n \in \mathbb{N}.$$

Then, for arbitrary $\mathbf{v} \in \mathbf{U}$,

$$
\begin{aligned}
|\langle \mathbf{F}(\theta_*, \Phi_*) - \mathbf{F}(\theta_n, \Phi_n), \mathbf{v}\rangle| &\le |\langle \mathbf{F}(\theta_*, \Phi_*) - \mathbf{F}(\theta_*, \Phi_n), \mathbf{v}\rangle| + |\langle \mathbf{F}(\theta_*, \Phi_n) - \mathbf{F}(\theta_n, \Phi_n), \mathbf{v}\rangle| \\
&\le 2\alpha_e|((\nabla^2\Phi_0\nabla(\Phi_* - \Phi_n))\theta_*, \mathbf{v})| + 2\alpha_e|((\nabla^2\Phi_0\nabla\Phi_n)(\theta_* - \theta_n), \mathbf{v})| \\
&+ \alpha_g|((\theta_* - \theta_n)\mathbf{g}, \mathbf{v})| \\
&\le C|((\nabla^2\Phi_0\nabla(\Phi_* - \Phi_n))\theta_*, \mathbf{v})| + CK\|\theta_* - \theta_n\|_4\|\mathbf{v}\|_6 + C\|\theta_* - \theta_n\|_4\|\mathbf{v}\|_{\frac{4}{3}}.
\end{aligned}
$$

Noting that $\theta_*\, \mathbf{v} \cdot \nabla^2\Phi_0 \in L^2(\Omega)$ follows by the assumption on Φ_0 and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, the first term converges to 0 due to $\Phi_n \rightharpoonup \Phi_*$ in H^1. Moreover, both other terms converge to 0 by $\theta_n \to \theta_*$ in L^4. □

A justification for the proposed approximations can be given by employing Lemma 2.13. Therefore, let $\mathbf{F}^*(\theta, \Phi) := \langle (\nabla\Phi)^2\, \nabla\theta, \cdot\rangle_{\mathbf{U}^*}$ and assume that a solution $(\mathbf{u}^*, \theta^*, \Phi^*)$ of Problem 3.1 with $\mathbf{F} = \mathbf{F}^*$ exists with $\Phi^* \in W^{1,6}(\Omega)$. Now, let $(\mathbf{u}, \theta, \Phi)$ denote another solution for $\mathbf{F} = \mathbf{F}_{s,0}$. If either $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ are fixed of if the data is sufficiently small, then Lemma 2.13 yields

$$
\|\nabla(\mathbf{u}^* - \mathbf{u})\| + \|\nabla(\theta^* - \theta)\| \le C\, D_{\mathbf{F}},
$$

with

$$
\begin{aligned}
D_{\mathbf{F}} &= \sup_{w\in\mathbf{V}}\sup_{\theta\in\Theta} \frac{|\langle \mathbf{F}^*(\theta + \theta_b, \Phi^* + \Phi_b) - \mathbf{F}_{s,0}(\theta + \theta_b, \Phi + \Phi_b), w\rangle_{\mathbf{U}^*}|}{\|\nabla w\|\|\theta + \theta_b\|_{1,2}} \\
&\le C\|\nabla(\Phi^* - \Phi_0)\|_6\left(\|\nabla(\Phi^* + \Phi_b)\|_6 + \|\nabla(\Phi_0 + \Phi_b)\|_6\right).
\end{aligned}
$$

In this case, the difference between both solutions is proportional to the difference between the exact potential Φ^* and its approximation Φ_0. Such an a priori approximation Φ_0 could be defined as regularization of the solution Φ_{00} of Gauss's law for some given reference temperature θ_0, i.e.

$$
(\epsilon(\theta_0)\nabla(\Phi_{00} + \Phi_b), \nabla\beta) = 0 \text{ for all } \beta \in \Upsilon.
$$

Note that

$$
\|\nabla(\Phi_1 - \Phi_2)\| \le \frac{\|\epsilon^{(1)} - \epsilon^{(2)}\|_\infty}{\epsilon_-^{(2)}}\|\nabla(\Phi_1 + \Phi_b)\|
$$

for potentials Φ_i solving Gauss's law with respective permittivities $\epsilon^{(i)} \in L^\infty(\Omega)$, $\epsilon^{(i)} \ge \epsilon_-^{(i)} > 0$ a.e. If the permittivity is chosen as in [20], i.e. $\epsilon(\theta) = \epsilon_r(1 - \gamma\theta)$, we obtain for the relative H^1-deviation between Φ_{00} and the potential Φ^* determined by the correct temperature $\theta^* + \theta_b$:

$$
\frac{\|\nabla(\Phi_{00} - \Phi^*)\|}{\|\nabla(\Phi^* + \Phi_b)\|} \le \frac{\gamma\|\theta_0 - (\theta^* + \theta_b)\|_\infty}{1 - \gamma\theta_\infty}. \tag{3.7}
$$

Here, we assumed that both temperatures satisfy the same maximum principle, thus having their values a.e. in an interval of width $d\theta = \sup_\Omega \theta_0 - \inf_\Omega \theta_0 = \sup_\Omega(\theta^* + \theta_b) - \inf_\Omega(\theta^* + \theta_b)$. When considering dielectric fluids with permittivity of low temperature dependency, e.g. some silicon oils take values $\gamma = \mathcal{O}(10^{-3})$, and temperature regimes in which the Boussinesq approximation is fairly accurate, i.e. $d\theta, \theta_\infty = \mathcal{O}(1)$, then the right hand side term in (3.7) is rather small. If, in addition, the effect of regularization is moderate, i.e. $\|\nabla(\Phi_{00} - \Phi_0)\|$ is small, we heuristically conclude that the proposed modellization of the DEP force is justified by the underlying physics under some restrictions on the fluid and the temperature boundary conditions. In a work that is currently under preparation, we will substantiate this reasoning with numerical experiments.

Following the previously stated idea of regularization, we propose another approximation to $\mathbf{f}_{E,a}$ that is based on mollification of the electric potential.

Definition 3.12. *(Mollifier)*
A nonnegative function $\psi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ *with* $\operatorname{supp}(\psi) = \overline{B_1(0, \mathbb{R}^d)}$ *and* $\int_{\mathbb{R}^d} \psi(x)dx = 1$ *is called* mollifier. *For such kind of function and* $t > 0$ *define the* mollifying operator

$$\begin{aligned} S_{\psi,t} \colon L^1(\Omega) &\to C_c^\infty(\mathbb{R}^d) \\ f &\mapsto f_t := (\tilde{f} * \psi_t) = \int_{\mathbb{R}^d} \tilde{f}(\cdot - y)\psi_t(y)dy \end{aligned}$$

with $\psi_t(y) := t^{-d}\psi(\frac{y}{t})$ *and* $\tilde{f}$ *denotes the extension of* f *by 0 outside of* Ω.

By replacing Φ by $S_{\psi,t}\Phi$ in the formulation of $\mathbf{f}_{E,a}$, we are able to construct a body force $\mathbf{F} = \mathbf{F}_t$ that satisfies Assumption 2.9, thus implying existence of a family of solutions of Problem 3.1, $\{\mathbf{u}_t, \theta_t, \Phi_t\}_{t>0}$. Since $S_{\psi,t}$ converges point wise to the identity operator on $L^p(\Omega)$ as $t \to 0$, see Lemma A.9, it is natural to ask whether a sequence of solutions $(\mathbf{u}_{t_n}, \theta_{t_n}, \Phi_{t_n})_n$ with $t_n \to 0$ converges in some sense. To answer this question, we need to introduce another modification of $\mathbf{f}_{E,a}$ to ensure that the growth parameters, $a_\mathbf{F} = a_\mathbf{F}(t)$ and $b_\mathbf{F} = b_\mathbf{F}(t)$, stay bounded for $t \to 0$.

Definition 3.13. *(Cut off operator)*
For $K > 0$ *let* $m_K \in L^\infty(\mathbb{R}^d)^d$ *denote a Lipschitz continuous function with* $m_K(x) = x$ *if* $|x| \leq K$. *Define the cut-off operator*

$$\begin{aligned} P_{m_K} \colon L^1(\Omega)^d &\to L^\infty(\Omega)^d \\ g &\mapsto m_K \circ g. \end{aligned}$$

By combining the previously defined operators, we may define a regularized electric gravity $\mathbf{g}_{E,t}$.

Definition 3.14. *(Regularized electric gravity and DEP force)*
Let a mollifier ψ *and a cut off function* m_K *according to Definitions 3.12 and 3.13 be given. For* $t > 0$ *define the regularized electric gravity*

$$\begin{aligned} \mathbf{g}_{E,t} \colon H^1(\Omega) &\to L^3(\Omega)^d \\ \Phi &\mapsto P_{m_K}\left[\nabla^2 S_{\psi,t}\Phi \cdot \nabla S_{\psi,t}\Phi\right]. \end{aligned}$$

The corresponding body force is defined by

$$\begin{aligned} \mathbf{F}_t \colon H^1(\Omega) \times H^1(\Omega) &\to \mathbf{U}^* \\ (\theta, \Phi) &\mapsto -2\alpha_e(\theta\, \mathbf{g}_{E,t}[\Phi], \cdot) - \alpha_g(\theta\, \mathbf{g}, \cdot). \end{aligned}$$

Proposition 3.15. *(Properties of mollified body force)*
$\mathbf{F}_t$ *satisfies Assumption 2.9 with growth rates* $a_\mathbf{F}, b_\mathbf{F}$ *being independent of* t.

Proof. For $\Phi \in H^1(\Omega)$, note that the following estimates hold by means of Lemma A.9 and Definition 3.13

$$\begin{aligned} \|\nabla S_{\psi,t}\Phi\|_6 &\leq C\|\nabla S_{\psi,t}\Phi\|_\infty \leq CC_\nabla(\psi, t)\|\Phi\|_2 \leq C_\nabla(t)\|\Phi\|_{1,2} \\ \|\nabla^2 S_{\psi,t}\Phi\|_6 &\leq C\|\nabla^2 S_{\psi,t}\Phi\|_\infty \leq CC_{\nabla^2}(\psi, t)\|\Phi\|_2 \leq C_{\nabla^2}(t)\|\Phi\|_{1,2} \\ \|\mathbf{g}_{E,t}[\Phi]\|_3 &\leq C\|m_K\|_\infty \\ \|P_{m_K}[f_1] - P_{m_K}[f_2]\|_p &\leq L_{m_K}\|f_1 - f_2\|_p \end{aligned} \tag{3.8}$$

with L_{m_K} denoting the Lipschitz constant of m_K and C denoting a generic constant that only depends on Ω and p.

We abbreviate the proof by setting the involved physical parameters to 1 and only considering the DEP term of $\mathbf{F}_t$, since the stated assertions easily follow for $(\theta\mathbf{g}, \cdot)$. Assertion (i) of Assumption 2.9 follows from the estimates

$$\begin{aligned}|\langle \mathbf{F}_t(\theta_1, \Phi) - \mathbf{F}_{\psi,t}(\theta_2, \Phi), \mathbf{v}\rangle| &= |((\theta_1 - \theta_2)\mathbf{g}_{E,t}[\Phi], \mathbf{v})| \\ &\leq \|\theta_1 - \theta_2\|_2 \|\mathbf{g}_{E,t}[\Phi]\|_3 \|\mathbf{v}\|_6 \\ &\leq C\|m_K\|_\infty \|\theta_1 - \theta_2\|_{1,2} M_6 \|\nabla \mathbf{v}\|\end{aligned}$$

and

$$\begin{aligned}|\langle \mathbf{F}_t(\theta, \Phi_1) - \mathbf{F}_t(\theta, \Phi_2), \mathbf{v}\rangle| &= |(\theta\,(\mathbf{g}_{E,t}[\Phi_1] - \mathbf{g}_{E,t}[\Phi_2]), \mathbf{v})| \\ &\leq \|\theta\|\|\mathbf{v}\|_6 \|\mathbf{g}_{E,t}[\Phi_1] - \mathbf{g}_{E,t}[\Phi_2]\|_3 \\ &\leq M_6 L_{m_K} \|\theta\|_{1,2} \|\nabla \mathbf{v}\| \\ &\quad \left(\|\nabla^2 S_{\psi,t}(\Phi_1 - \Phi_2)\|_6 \|\nabla S_{\psi,t}\Phi_1\|_6 + \|\nabla^2 S_{\psi,t}(\Phi_2)\|_6 \|\nabla S_{\psi,t}(\Phi_1 - \Phi_2)\|_6\right)\end{aligned}$$

for arbitrary $R > 0$, $\Phi_i, \theta_i \in B_R(0, H^1(\Omega))$, $\Phi, \theta \in H^1(\Omega), \mathbf{v} \in \mathbf{U}$ and by using (3.8).

Assertion (ii) follows from

$$|\langle \mathbf{F}_t(\theta, \Phi), \mathbf{v}\rangle| = |(\theta\mathbf{g}_{E,t}[\Phi], \mathbf{v})| \leq \|\mathbf{g}_{E,t}[\Phi]\|_3 \|\theta\|_2 \|\mathbf{v}\|_6 \leq C\|m_K\|_\infty \|\theta\|_{1,2} \|\mathbf{v}\|_6,$$

i.e. $b_{\mathbf{F}} = 0$ and $a_{\mathbf{F}} = C\|m_K\|_\infty$.

Finally, let sequences $(\theta_n)_n, (\Phi_n)_n$ be given according to Assumption 2.9 (iii). Then,

$$\begin{aligned}|\langle \mathbf{F}_t(\theta_*, \Phi_*) - \mathbf{F}_t(\theta_n, \Phi_n), \mathbf{v}\rangle| &\leq |((\theta_* - \theta_n)\mathbf{g}_{E,t}[\Phi_*], \mathbf{v})| + |(\theta_n\,(\mathbf{g}_{E,t}[\Phi_*] - \mathbf{g}_{E,t}[\Phi_n])\,, \mathbf{v})| \\ &\leq C\|m_K\|_\infty \|\theta_* - \theta_n\|_4 \|\mathbf{v}\|_6 + CL_{m_K} \|\theta_n\| \|\mathbf{v}\|_6 \\ &\quad \left(\|\nabla^2 S_{\psi,t}(\Phi_* - \Phi_n)\|_6 \|\nabla S_{\psi,t}\Phi_*\|_6 + \|\nabla^2 S_{\psi,t}(\Phi_n)\|_6 \|\nabla S_{\psi,t}(\Phi_* - \Phi_n)\|_6\right) \\ &\to 0\end{aligned}$$

Here, convergence follows from $\|\Phi_n - \Phi_*\|_4 \to 0$, $\|\theta_n - \theta_*\|_4 \to 0$, the uniform boundedness of $\|\Phi_n\|_{1,2}$, $\|\theta_n\|_{1,2}$ and the estimates (3.8). □

Due to Proposition 3.15, $\mathbf{F}_t$ satisfies the requirements of the existence Theorem 3.5 for all $t > 0$. Under the remaining conditions of 3.5, we may therefore state the existence of a family of solutions $\{\mathbf{u}_t, \theta_t, \Phi_t\}$ of Problem 3.1. Moreover, Theorem 3.5 provides energy bounds

$$\|\nabla \mathbf{u}_t\| \leq G_{\mathbf{u}}, \quad \|\nabla \theta_t\| \leq G_\theta, \quad \|\nabla \Phi_t\| \leq G_\Phi$$

with constants G_i that depend on $\mathbf{F}_t$ only via $a_{\mathbf{F}}(t) = \text{const}$, $b_{\mathbf{F}} = 0$. Thus, they are uniform w.r.t. t. Choosing an arbitrary sequence $t_n \to 0$ and using again the reflexivity of $\mathbf{U}, \Theta, \Upsilon$, we obtain functions $\mathbf{u}_* \in \mathbf{U}$, $\theta_* \in \Theta$, $\Phi_* \in \Upsilon$ such that

$$\mathbf{u}_{t_n} \rightharpoonup \mathbf{u}_*, \quad \theta_{t_n} \rightharpoonup \theta_*, \quad \Phi_{t_n} \rightharpoonup \Phi_*.$$

Moreover, we have by construction, $\|\mathbf{g}_{E,t_n}[\Phi_{t_n}]\|_3 \leq C\|m_K\|_\infty$. Since $L^3(\Omega)^d$ is reflexive, we also obtain

$$\mathbf{g}_{E,n} := \mathbf{g}_{E,t_n}[\Phi_{t_n}] \rightharpoonup \mathbf{g}_E \text{ for some } \mathbf{g}_E \in L^3(\Omega)^d.$$

Letting $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$ denote arbitrary test functions for Problem 3.1, we obtain convergence of all bi- and trilinear forms as in the proof of Theorem 3.5, e.g. $a_{\mathbf{v}}(\mathbf{u}_{t_n}, \mathbf{v}) \to a_{\mathbf{v}}(\mathbf{u}_*, \mathbf{v})$, etc. Thus, it only remains to consider convergence of the term $\langle \mathbf{F}_{t_n}(\theta_{t_n}, \Phi_{t_n}), \mathbf{v}\rangle_{\mathbf{U}^*}$. To this end,

$$\begin{aligned}|(\theta_n \mathbf{g}_{E,n}, \mathbf{v}) - (\theta_* \mathbf{g}_{E,*}, \mathbf{v})| &\leq |(\theta_*(\mathbf{g}_{E,*} - \mathbf{g}_{E,n}), \mathbf{v})| + |((\theta_n - \theta_*)\mathbf{g}_{E,n}, \mathbf{v})| \\ &\leq |(\theta_*(\mathbf{g}_{E,*} - \mathbf{g}_{E,n}), \mathbf{v})| + C\|\theta_n - \theta_*\|_4 \|\mathbf{g}_{E,n}\|_3 \|\mathbf{v}\|_6 \\ &\to 0\end{aligned}$$

Here, the first term converges towards 0 by $\mathbf{g}_{E,n} \rightharpoonup \mathbf{g}_E$, and the second term by uniform L^3 boundedness of $\mathbf{g}_{E,n}$ and the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$.

These considerations are summarized by the following theorem.

Theorem 3.16. *(Existence of solutions with weak approximation of electric gravity)*
Let the requirements of Theorem 3.5 and Proposition 3.15 hold. Then, there are $(\mathbf{u}, \theta, \Phi, \mathbf{g}_E) \in \mathbf{V} \times \Theta \times \Upsilon \times L^3(\Omega)^d$ *such that*

$$\begin{aligned} a_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) + c_v(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) &= -((\theta + \theta_b)(2\alpha_e \mathbf{g}_E + \alpha_g \mathbf{g}), \mathbf{v}) + \langle \mathbf{f}_{\mathbf{v}}, \mathbf{v} \rangle_{\mathbf{U}^*} \\ a_\tau(\theta + \theta_b, \tau) + c_\tau(\tilde{\mathbf{u}}, \theta + \theta_b, \tau) &= \langle f_\tau, \tau \rangle_{\Theta^*} \\ a_\beta(\bar{\theta} + \theta_b, \Phi + \Phi_b, \beta) &= \langle f_\beta(\theta + \theta_b), \beta \rangle_{\Upsilon^*} \end{aligned} \tag{3.9}$$

holds for all $(\mathbf{v}, \tau, \beta) \in \mathbf{V} \times \Theta \times \Upsilon$. *The connection between* $\mathbf{g}_E$ *and* Φ *is given in the sense that there is a sequence* $(t_n, \Phi_n)_n \subset (0, \infty) \times \Upsilon$ *with*

$$\begin{aligned} t_n &\to 0 \\ \Phi_n &\rightharpoonup \Phi \text{ in } H^1(\Omega) \\ \mathbf{g}_{E,t_n}[\Phi_n] &\rightharpoonup \mathbf{g}_E \text{ in } L^3(\Omega). \end{aligned} \tag{3.10}$$

with approximate electric gravity given by Definition 3.14.

According to Theorem 3.16, we obtain a notion of a solution for the stationary TEHD equations 3.1, where the strong connection between electric gravity and potential, $\mathbf{g}_E = \nabla^2\Phi \cdot \nabla\Phi$, is replaced by the weaker form (3.10).

4 Conclusion and Outlook

In this work, we proposed a functional analytic framework that allows to prove existence, stability and uniqueness of solutions to the stationary TEHD equations. In doing so, we additionally extended the theory on well-posedness of the standard stationary Boussinesq equations by allowing a more general force term. Due to the high regularity requirements imposed by the DEP force, it was necessary to replace it by a suitable approximation. So far, this approximation has been justified by heuristic arguments. However, we have already conducted numerical experiments that underline this reasoning. These results will be presented in a further publication, together with a numerical analysis of the Finite Element Method applied to the stationary TEHD equations. An extension of the presented results to the instationary problem is part of our ongoing research.

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A Appendix

Lemma A.1. *(Subspaces of Separable Spaces)*
Let X *denote a separable normed space. Let* $U \subset X, U \neq \emptyset$ *denote some subspace. Then* U *is separable.*

Proof. Let $Q \subset X$ denote a countable set with $\overline{Q} = X$. Let $\{c_n : n \in \mathbb{N}\} = \mathbb{Q}_{>0}$ denote an enumeration of the positive rational numbers. For each $q \in Q, n \in \mathbb{N}$ choose $u_n^q \in U \cap B_X(q, c_n) =: U_{q,n}$ if the intersection is non-empty. Then,

$$U' := \bigcup_{n \in \mathbb{N}, q \in Q, U_{q,n} \neq \emptyset} \{u_n^q\}$$

is a countable subset of U.

Now, let $u \in U \subset X$, $\epsilon > 0$ be arbitrary. Since Q is dense in X, there exists $q \in Q$ with $\|u - q\| < \frac{\epsilon}{4}$. Choose a positive rational number c_N such that $\frac{\epsilon}{4} \leq c_N \leq \frac{\epsilon}{2}$. Then, $\|u - q\| < c_N$ and thus, $u \in U_{q,N}$, i.e. $U_{q,N} \neq \emptyset$. Therefore, there exists an element $u_* = u_N^q \in U'$, which satisfies $\|u_* - q\| < c_N \leq \frac{\epsilon}{2}$. Therefore,

$$\|u_* - u\| \leq \|u_* - q\| + \|q - u\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.$$

Since u and ϵ are chosen arbitrarily, U' is dense in U. By construction, U' is a countable subset of U and therefore the assertion follows. □

Lemma A.2. *(Compact Operator in Finite Dimensions)*
Let X denote a finite dimensional Hilbert space and $F\colon X \to X$ a continuous operator. Then, F is compact.

Proof. Let $M \subset X$ denote a bounded set. Then, $\overline{M}$ is closed and bounded, thus compact since X is finite dimensional. Let $K := F(\overline{M})$ and $\mathcal{C}$ denote an open cover of K. Since F is continuous, $F^{-1}(U)$ is open for all open subset $U \subset X$. Therefore, $\mathcal{D} := \{F^{-1}(U) : U \in \mathcal{C}\}$ is an open cover of $\overline{M}$. Since $\overline{M}$ is compact, there exists $U_1, \ldots, U_m \in \mathcal{C}$ for some $m \geq 1$ such that $\overline{M} \subset F^{-1}(U_1) \cup \ldots \cup F^{-1}(U_m)$. Now

$$\begin{aligned} F(\overline{M}) &\subset F\left(F^{-1}(U_1) \cup \ldots \cup F^{-1}(U_m)\right) \\ &\subset F(F^{-1}(U_1)) \cup \ldots \cup F(F^{-1}(U_m)) \\ &= U_1 \cup \ldots \cup U_m. \end{aligned}$$

Therefore, $\mathcal{C}$ contains a finite covering of $F(\overline{M})$, i.e. $F(\overline{M})$ is compact. Thus, it is also bounded, implying that $F(M) \subset F(\overline{M})$ is bounded as well. Therefore, the closure of $F(M)$ is bounded and closed, thus compact in X. □

Theorem A.3. *(Lax-Milgram Lemma, Satz 4.2 in [3])*
Let H denote a real Hilbert space with norm $\|\cdot\|_H$, $a\colon H \times H \to \mathbb{R}$ a bilinear form and $l\colon H \to \mathbb{R}$ a linear form. Assume there exists $M, N, \alpha > 0$ such that for all $u, v \in H$:

$$\begin{aligned} a(u, v) &\leq M\|u\|_H \|v\|_H \\ a(v, v) &\geq \alpha \|v\|_H^2 \\ l(v) &\leq N\|v\|_H \end{aligned}$$

Then, there exists a unique solution u of

$$a(u, v) = l(v) \text{ for all } v \in H. \tag{A.1}$$

Moreover, this solution satisfies

$$\|u\|_H \leq \frac{N}{\alpha}$$

Theorem A.4. *(Well-Posedness under inf-sup Condition, Satz 2.1 in [21])*
Let $(X, \|\cdot\|_X)$ and $(M, \|\cdot\|_M)$ denote Hilbert spaces and let a bilinear form $b\colon X \times M \to \mathbb{R}$ be given which is assumed to be bounded, i.e.

$$|b(v,p)| \leq M\|v\|_X\|p\|_M \text{ for all } v \in X\ p \in M.$$

Define linear operators

$$B\colon X \to M^*,\ x \mapsto b(x,\cdot)$$
$$B^*\colon M \to X^*,\ p \mapsto b(\cdot,p)$$

and spaces

$$V := \{v \in X :\ b(v,q) = 0\ \forall q \in M\}$$
$$V^\circ := \{g^* \in X^* :\ \langle g, v\rangle_X = 0\ \forall v \in V\}$$
$$V^\perp := \{x \in X :\ (x,v)_X = 0\ \forall v \in V\}$$

Then, the following assertions are equivalent.

(i) There is $\beta > 0$ such that

$$\inf_{p\in M, p\neq 0}\ \sup_{v \in X, v\neq 0} \frac{b(v,p)}{\|v\|_X\|p\|_M} \geq \beta.$$

(ii) B^* is an isomorphism from M to V° with

$$\|B^*p\|_{X^*} \geq \beta\|p\|_M \text{ for all } p \in M.$$

(iii) B is an isomorphism from $V^\perp$ to M^* with

$$\|Bv\|_{M^*} \geq \beta\|v\|_X \text{ for all } v \in V^\perp.$$

Theorem A.5. *(Poincare-Friedrichs inequality for H_0^1, Theorem I.1.1 in [11])*
Let Ω be open, bounded and connected. Then, there is $C_{PF} > 0$ such that

$$\|u\|_2 \leq C_{PF}\|\nabla u\|_2 \text{ for all } u \in H_0^1(\Omega).$$

Theorem A.6. *(Generalized Poincare-Friedrichs, Proposition 7.1 in [7])*
Let Ω be open, bounded and connected and $p \in (1,\infty)$. Assume that $(X, \|\cdot\|_X)$ is a closed subspace of $W^{1,p}(\Omega)$ that does not contain the function $f \equiv 1$ and for which the restriction of the canonical embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ to X is compact. Then, $\|\cdot\|_X$ is equivalent to $\|\nabla\cdot\|_p$.

Theorem A.7. *(Poincare-Friedrichs inequality for $W_D^{1,2}(\Omega)$)*
Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Assume that $\partial\Omega = \Gamma_N + \Gamma_D$ with Γ_D having positive $(d-1)-$Hausdorff measure. Then, there is $C_{PF} > 0$ such that

$$\|u\|_2 \leq C_{PF}\|\nabla u\|_2 \text{ for all } u \in H_D^1(\Omega).$$

Proof. Let $(X, \|\cdot\|_X) := (H_D^1(\Omega), \|\cdot\|_{1,2})$. By definition, $X = \overline{C_D^{0,1}(\overline{\Omega}) \cap W^{1,6}(\Omega)}^{W^{1,2}}$, and $\gamma_D v = v_{|\Gamma_D}$ for all $v \in C^{0,1}(\overline{\Omega})$ according to Theorem 1.5.1.3 in [12] with $\gamma_D \in \mathcal{L}(W^{1,2}(\Omega), W^{\frac{1}{2},2}(\Gamma_D))$ denoting the boundary trace operator w.r.t. Γ_D. Since $C_D^{0,1}(\overline{\Omega}) \subset \ker(\gamma_D)$, we have $X \subset \ker(\gamma_D)$ by continuity of γ_D.

Moreover, we have the embeddings

$$(H^1_D(\Omega), \|\cdot\|_{1,2}) \hookrightarrow W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$

where the first embedding is obvious and the second embedding follows from the Sobolev embedding Theorem 4.12 in [1]. Therefore,

$$(H^1_D(\Omega), \|\cdot\|_{1,2}) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Finally, the function $f\colon \Omega \to \mathbb{R},\ x \mapsto 1$ is not contained in X, since $\gamma_D(f) \equiv 1 \neq 0$ on Γ_D and Γ_D has non-zero measure. Thus Theorem A.6 yields the existence of some $C > 0$ such that

$$\|\cdot\|_2 \leq \|\cdot\|_{1,2} := \|\cdot\|_X \leq C\|\nabla\cdot\|_2 \text{ on } X.$$

□

Lemma A.8. *(Properties of convolution)*
Let $\phi \in C_0^\infty(\mathbb{R}^d)$, $f \in L^1_{loc}(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Then,

(i) $\|(g * \phi)\|_\infty \leq \|g\|_p \|\phi\|_{p^*}$

(ii) $\frac{\partial}{\partial x_i}(f * \phi) = (f * \frac{\partial}{\partial x_i}\phi)$

(iii) $\frac{\partial^2}{\partial x_i \partial x_j}(f * \phi) = (f * \frac{\partial^2}{\partial x_i \partial x_j}\phi)$

Proof. (i) follows from

$$\|(g * \phi)\|_\infty = \sup_{x \in \mathbb{R}^d} |\int_{\mathbb{R}^d} g(x-y)\phi(y)dy| = \sup_{x \in \mathbb{R}^d} |\int_{\mathbb{R}^d} g(y)\phi(x-y)dy|$$
$$\leq \sup_{x \in \mathbb{R}^d} \|g\|_p \|\phi(x-\cdot)\|_{p^*} = \|g\|_p \|\phi\|_{p^*}$$

(ii) follows by applying the definition of ∂_i, the Mean Value Theorem and Dominated Convergence Theorem. (iii) follows by iteratively applying (ii). □

Lemma A.9. *(Properties of mollifiers)*
Let $p \in [1,\infty)$ and p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$ and $f \in L^p(\Omega)$. The following properties hold for the Mollifier operator $S_{\psi,t}$ given by Definition 3.12.

(i) $S_{\psi,t} \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ *with* $\|S_{\psi,t}\| \leq \|\psi_t\|_{p^*,\mathbb{R}^d}$

(ii) $S_{\psi,t}f \to f$ *in* L^p *for* $t \to 0$.

(iii) $\|\nabla S_{\psi,t}f\|_\infty \leq C_\nabla(\psi,t)\|f\|_p$

(iv) $\|\nabla^2 S_{\psi,t}f\|_\infty \leq C_{\nabla^2}(\psi,t)\|f\|_p$

(v) *If* $f_n \to f$ *in* L^p, *then* $\nabla S_{\psi,t}f_n \to \nabla S_{\psi,t}f$ *in* $L^\infty(\Omega)$

Proof. (i): Let $g \in L^p(\Omega)$ with $\|g\|_p = 1$ and denote by $\tilde{g}$ its extension by 0 outside of Ω. Then, by Lemma A.8 (i),

$$\|S_{\psi,t}g\|_\infty = \|(\tilde{g} * \psi_t)\|_\infty \leq \|\tilde{g}\|_{p,\mathbb{R}^d}\|\psi_t\|_{p^*,\mathbb{R}^d} = \|\psi_t\|_{p^*,\mathbb{R}^d}.$$

Since $S_{\psi,t}$ is obviously linear, (i) follows.

For (ii), see e.g. Theorem 2.29 in [2]. (iii) is obtained by using Lemma A.8 in

$$\begin{aligned}
\|\nabla S_{\psi,t}f\|_\infty = \sup_{x\in\Omega}\left(\sum_{i=1}^{d}(|\partial_i S_{\psi,t}f(x)|)^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{d}(\sup_{x\in\Omega}|\partial_i S_{\psi,t}f(x)|)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{d}(\|\partial_i(\tilde{f}*\psi_t)\|_\infty)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{d}(\|f\|_p\|\partial_i\psi_t\|_{p^*,\mathbb{R}^d})^2\right)^{\frac{1}{2}} \\
&=: C_\nabla(\psi,t)\|f\|_p.
\end{aligned}$$

(iv) follows analogously. (v) is a direct consequence of (iii) and the linearity of $S_{\psi,t}$. □

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