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Notes on Mathematical and Mensurational Papyri

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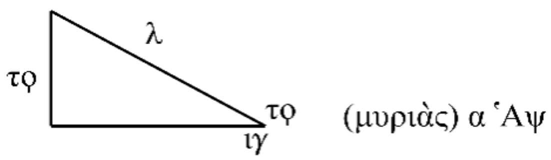
1. P.Math. B recto, problem b2.

§1 The first line of the problem (line 4 of the papyrus sheet) is printed and translated in the edition as [χωρ]ί̣ον τετράγωνον εἰς ὃ πλ[ευρὸν ἔχον], ‘a rectangular field having its side’. The editors divide the letters to form the phrase εἰς ὃ πλ[ευρόν], but what is intended is surely εἰσόπλ[ευρον], *l.* ἰσόπλευρον, ‘equilateral’, which qualifies the word for quadrilateral, τετράγωνον. The field (χωρίον) has the shape of an equilateral quadrilateral, thus a square (so the editors based on computed sides[1]).

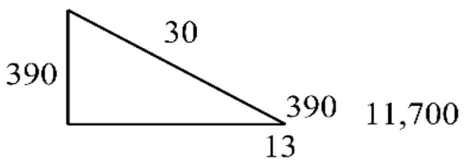
2. P.Math. B verso, problem b4.

§2 Precisely the same qualifier, ἰσόπλευρον, ‘equilateral’, is to be read in problem b4 on B verso. The editors give the following text and translation:

[τετράγ]ω̣νον εἰς ὃ πλευρ̣ὸν ἔχο[ν ἀπὸ νότου]
[εἰς βορρ]ᾶς σχοινίων λ̅, καὶ βάσις [σχοινί-]
[ων ι̅γ̅. ε]ὑ̣ρεῖν τὸ ἐνβαδόν. οὕτω μ[- - - τὸ λ̅]
[ἐφ’ ἑα]υτά. γί(νεται) Ͳ̅. ἐπὶ τὸν ι̅γ̅. γί(νεται) (μυριὰς) α̅[͵Α̅ψ̅. παρὰ]
[τὸν λ̅]. γί(νεται) τ̅ϙ̣̅. οὕτως ἔχει ὁμοίω̣ς.
diagram λ | τϙ | τϙ | ιγ | (μυριὰς) α ͵Αψ



A rectangle having its side [from south]
[to] north 30 schoinia, and base [13 schoinia.]
To find the area. I? proceed? as follows. … [The 30]
[times] itself is 900. Times 13. The result is 1[1,700. (I divide) by]
[the 30.] The result is 390. This way for similar cases.
diagram



1 Bagnall and Jones 2019: 125.

§3 Since only [- - -]ω̣νον survives at the beginning of line 2, restoring [τρίγ]ω̣νον is equally possible, and reading [τρίγ]ω̣νον εἰσόπλευρ̣ον helps to account fully for the rest of the text. The task of the problem is to determine the area of an equilateral triangle, the side of which equals 30 schoinia. It is computed by squaring the side of the triangle, multiplying the result by 13 and dividing the product by 30. In modern notation, the area A = $a^2 \times \frac{13}{30}$, where *a* is the side of the triangle and $\frac{1}{3}\,\frac{1}{10}$ (= $\frac{13}{30}$) is a well attested approximation for $\frac{\sqrt{3}}{4}$ (see note to 5–6 below). My identification of the problem's subject matter leads me to suggest the following text:

[τρίγ]ω̣νον εἰσόπλευρ̣ον ἔχο[ν -ca.6-]
[πλευρ]ὰς σχοινίων λ̄, καὶ βάσις [-ca.6-]
[- - - ε]ὑ̣ρεῖν τὸ ἐνβαδόν. οὕτω μ̣ε[τρ- - - τὸ λ̄]
[ἐφ' ἑα]υτά· γί(νεται) ϡ̄. ἐπὶ τὸν ιγ̄· γί(νεται) (μυριὰς) ᾱ[͵Αψ̄. παρὰ]
[τὸν] λ̣̄· γί(νεται) τϙ̄. οὕτως ἔχει ὁμοίως̣.
diagram λ | τϙ | τϙ | ιγ | (μυριὰς) α ͵Αψ

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2 *l.* ἰσόπλευρον 4 *l.* ἐμβαδόν

'An equilateral triangle with the sides of 30 schoinia and the base [- - -]. To find its area. We/I/you measure it this way: multiply 30 by itself, the result is 900; times 13, the result is 11,700; divide by 30, the result is 390. This is how it is for similar cases.'

§4 3 [πλευρ]άς seems to me a plausible restoration, which may have been preceded by the article τάς at the end of line 2. The writer could have viewed the two sides opposite to the base as πλευραί and the base as βάσις, even though all three sides should theoretically be called πλευραί.

§5 βάσις [- - -]. One should not be surprised that the length of the base is stated even though the triangle is said to be equilateral. Since the numeral is lost, two possibilities for what could have been written there present themselves. It could have been 30, in which case the text of the problem was correctly transmitted; alternatively, if the writer misunderstood the problem, he may have given the length of the base as 13, as appears to be given in the drawing. I prefer not to restore the numeral here.

§6 4 μ̣ε[τρ- - -]. This is likely a form of μετρέω.

§7 5–6 The procedure employed here to determine the area of an equilateral triangle, in modern notation A = $a^2 \times \frac{13}{30}$, is used repeatedly in MPER I 1 recto (Soknopaiou Nesos, 1st c.; TM 63192),[2] where it is never explicitly formulated because it is taken for granted that the reader understands it. The description of the method, phrased very similarly to our papyrus, is found in pseudo-Heron, *Geometrica* 22.3 (Heiberg 1912: 392): τριγώνου ἰσοπλεύρου τὸ ἐμβαδὸν εὑρεῖν. τὴν πλευρὰν ἐφ' ἑαυτήν· ταῦτα ἐπὶ τὰ ιγ· ὧν λʹ ἔστω τὸ ἐμβαδόν, 'To find the area of an equilateral triangle. Multiply the side by itself; multiply the result by 13. The area will be 1/30th of that.'

§8 Diagram The drawings in P.Math. tend to be very schematic and consequently it is difficult to ascertain whether the 13 written by the 'base' of what looks like a right-angled triangle was meant to indicate its length. If it was, the writer likely failed to understand the problem perhaps because he was not familiar with the procedure for computing the area of an equilateral triangle. Most of the triangles in P.Math.,

2 For the discussion of the algorithm, cf. Bagnall and Jones 2019: 27–28, 45–49.

as in other contemporary collections of problems, are right-angled, which is exactly what he seems to have meant to depict.

3. P.Math. G recto, problem g2.

§9 The first line of the problem (line 22 on the sheet) is printed in the edition as ναύβιον στρονκύλουν (l. στρογγύλον) μισ̣ων[- - - περιφέρεια], and the problem is interpreted as seeking the volume of a trench in the form of a cylinder with a given circumference. On the image kindly provided to me by the editors it is possible to read the second word as μίου̣ρ̣ων, l. μύουρον or μείουρον, which means 'tapering'; cf. Heron, *Metrica* III 19 (Acerbi & Vitrac 2014: 348), where οἱ μείουροι is used as a collective term encompassing 'pyramids, cones and the like'. Since the excavation is circular and only one circumference is given, it must be conical. This explains the computation of its volume, which is based on equating it to the volume of a cylinder with a circumference half the circumference of the cone and the same height (lines 24–26). Although the procedure results in a seemingly absurd equating of the volume of a cone with 1/4 (instead of 1/3!) of the volume of a cylinder built on the same base, it is consistent with the general approach to measuring the volume of tapering solids by means of 'averaging approximation' which is employed in many problems in the codex and in other collections (cf. Bagnall and Jones 2019: 27–28). Since the cone has no top-face, the circumference of the cylinder to which it is approximated is simply taken to be twice smaller than that of the base of the cone.

4. P.Math. N recto, problems n2 and n3

§10 Problem n2 asks to find the capacity in artabas of a granary for which the dimensions of the length, width, and depth are given in lines 12–13: θησαυρὸς τρίγωνος, μῆκος πηχῶν κ̅, πλάτος πηχῶν| λ̣̅, βάθ̣ος πηχῶν ϛ̅. εὑρεῖν τὰς ἀρτάβας. The method of solving the problem consists of multiplying the width by the length, dividing the product by four and multiplying it by the depth before converting the last product to artabas. The editors interpret the shape of the granary as a triangular prism with a right-angled triangle as its base, the volume of which is erroneously computed as a product of a quarter—instead of a half—of the area and the height (Bagnall and Jones 2019: 147).

§11 Although the use of geometrical or mensurational terms in papyri is all but strict, sides of a triangle do not seem ever to be referred to by length and width, yet the diagram accompanying the problem presents a triangle. The key to understanding the described shape is furnished by a problem transmitted in the treatise *On Measuring All Sorts of Timber*, ascribed to Didymus of Alexandria, which features a similar description in the statement of one problem: ἔστω ξύλον τρίγωνον τὸ μὲν μῆκος πήχεων κ̅δ̅, τὸ δὲ πλάτος δακτύλων ι̅β̅, τὸ δὲ πάχος δακτύλων ι̅·. εὑρεῖν αὐτοῦ τὸ στερεόν, Didymus 41 (Heiberg 1927: 16), 'Let there be a triangular piece of wood, with the length of 24 cubits, the width of 12 fingers, the thickness of 10 fingers. To find its volume.' The volume is then computed as the product of the width and the thickness, which is divided by four and then multiplied by the length, that is, in the exact same way as is in problem n2. In the two main manuscripts of Didymus the problem is accompanied by a drawing of a rectangular pyramid put out on its side, so that the dimension one would call the height of the pyramid looks like the length of the object. I think a rectangular pyramid is indeed the shape of the solids both in Didymus and in problem n2 of P.Math. (and probably n3, too; see below). Its designation as τρίγωνον must refer not to the base of the pyramid, but to its side-view, so to speak. This might explain why the granary in problem n2 is depicted as a right-angled triangle, unless the person making the drawing simply took the word τρίγωνον in the statement of the problem as his cue. That the volume of the pyramids in Didymus and P.Math. is computed as 1/4 (not 1/3!) of a prism of the same height built on the same base must be owed to the use of the same principle of ‘averaging approximation’ which we see at work in the computation of the volume of a cone in problem g2.

§12 Problem n3 is evidently similar to problem n2, but all the parameters necessary for ascertaining the computational procedure are lost. The editors emend the text on the assumption that both granaries have the shape of a triangular prism (Bagnall and Jones 2019: 147–148). As I am inclined to believe that both are rectangular pyramids, I suggest the following changes (in bold):

P.Math.

ἐπ[ὶ] τὸ πλάτος, κ̅ [ἐ]π̣ὶ τὸ̣ν ι̅. γί(νεται) σ̅. ὧ[ν **ἥμισυ ρ̅**.]
ταῦ̣τα ποιῶ ἐπὶ τὸ βάθος, πηχῶ[ν **γ̅**. γ(ίνεται) τ̅. . . .]

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ἐπ[ὶ] τὸ πλάτος, κ̅ [ἐ]π̣ὶ τὸ̣ν ι̅. γί(νεται) σ̅. ὧ[ν **τὸ d γ(ίνεται) ν̅**.]
ταῦ̣τα ποιῶ ἐπὶ τὸ βάθος, πηχῶ[ν **ϛ̅**. γ(ίνεται) τ̅. . . .]

5. P.Oxy. 66 4537

§13 The papyrus, which is dated to the late sixth or early seventh century, contains measurements for a newly excavated water installation called a νέος λάκκος. The excavation apparently consisted of two parts, the cistern itself (lines 7–10) and something called an *anabateria* (lines 11–16), possibly a water conduit leading up to the cistern. As was conventional when measuring earth, linear dimensions of the excavated objects were recorded in cubits and the volume in naubia. I reproduce the relevant lines of the papyrus:

ἄνω πλάτος πήχ(εις) κδ
κάτω πλάτος πήχ(εις) κβ
βάθος πήχ(εις) ϛ
  εἰς ναύει(α) πη ϛ´
(καὶ) τῆς ἀναβατηρ(ίας) οὕτ(ως)·
μῆκ(ος) π̣ήχ(εις) κζ
ἄνω πλάτος πήχ(εις) ι
κάτω πλάτος πήχ(εις) ϛ
βάθος πήχ(εις) γ
  εἰς ναύει(α) κδ

10, 16. *l.* ναύβι(α)

‘Upper width 24 cubits, lower width 22 cubits, depth 6 cubits; converted to naubia it is 88 1/6. And (the dimensions) of the *anabateria* are as follows: length 27 cubits, upper width 10 cubits, lower width 6 cubits, depth 3 cubits; converted to naubia it is 24’ (lines 7–16)

§14 As can be seen from the linear dimensions of the *anabateria*, it had the shape of a trapezoidal prism. Its volume was computed as the product of the length, the average of the upper and lower width and the depth, which was then converted to naubia through division by 27 (although not recorded on the papyrus, the calculations can be easily reconstructed: $27 \times \frac{10+6}{2} \times 3 \div 27 = 24$). It is not immediately clear, however, what the shape of the cistern was. Puzzled by the absence of a length measurement among its linear dimensions, the editor, A. Syrcou, deduced it by a roundabout calculation from the volume and the two linear dimensions given in the papyrus. On the assumption that the cistern had the shape of a rectangular prism, she calculated the length at 17 1/4 cubits, but found it odd that it turned out to be shorter than the width. There is no reason for concern, however, because the cistern must have been circular, i.e., a conical frustum, even though the writer of the papyrus chose to call the upper and lower diameters 'width'. That it indeed had the shape of a conical frustum is borne out by the given numerical values. Following the standard procedure,[3] the volume of a conical frustum with an upper diameter of 24 cubits, lower diameter of 22 cubits and depth of 6 cubits can be computed as $\frac{3}{4} \times (\frac{24+22}{2})^2 \times 6 = 2380\ 1/2$ (cubic) cubits, which, converted to naubia, yields exactly 88 1/6.

6. P.Oxy. 77 5125

§15 The papyrus, dated to the sixth century, contains measurements of a cleaned cistern. The dimensions are listed in a way similar to P.Oxy. 66 4537: The upper and lower width and the depth are recorded for the cistern, but, unlike P.Oxy. 66 4537, no volume is computed. The editor, C. Luz, leaves open the possibility that the measurements are lacking the length, but also reports a suggestion of F. Morelli that the cistern has the shape of a truncated cone and that the width corresponds to the diameter. The latter suggestion is undoubtedly correct, but the computation based on it is misleading. On the assumption that a cubit equals 52.5 cm and $\pi \approx 3.14$ the volume of the cistern is computed as 98.37 m^3, which is then converted to c. 25 1/8 naubia. The cubit of 52.5 cm is the Egyptian cubit of seven palms, which would not have been in use at the time this text was written, nor would the value of π have been taken as anything but 3 in a mensurational text.[4] Leaving the issue of converting to modern units aside, the volume of a cistern with an upper diameter of 13 cubits, a lower of 11 cubits and a depth of 6 cubits can be computed following the usual procedure described in the note to P.Oxy. 66 4537, namely $\frac{3}{4} \times (\frac{11+13}{2})^2 \times 6 \div 27 = 24$ naubia.

Bibliography

Acerbi, F. and Vitrac, B. (2014) Héron d'Alexandrie. Metrica: Introduction, texte critique, traduction française et notes de commentaire. Pisa.

Bagnall, R.S. and Jones, A. (2019) Mathematics, Metrology, and Model Contracts. A Codex from Late Antique Business Education (P.Math.). New York.

Heiberg, J. (1912) Heronis Alexandrini Opera quae Supersunt Omnia. Vol. 4, Leipzig.

Heiberg, J. (1927) Mathematici graeci minores. I. Didymus, Copenhagen,1–23.

3 This procedure is detailed in 'S3A. Conic frustum volume algorithm A' in Bagnall and Jones 2019: 42–43.

4 From the Hellenistic period on, the cubit of six palms is generally used as the basic unit of length. Cf., among others, K. Maresch in the appendix 'Beobachtungen zu den Längen- und Flächenmassen Ägyptens in römischer und byzantinischer Zeit' in P. Koeln 7: pp. 177–187.

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