PYLON

**Pylon** 2 (2022)
ISSN: 2751-4722

# P.Math. Leaf A Verso, Mathematical Problem a3 Revisited: A New Algorithm in Greek Mensurational Mathematics

Roger S. Bagnall, Alexander Jones

Heidelberg: Propylaeum, 2022



DOI: https://doi.org/10.48631/pylon.2022.2.92969

Citation:
R.S. Bagnall, A. Jones, "P.Math. Leaf A Verso, Mathematical Problem a3 Revisited: A New Algorithm in Greek Mensurational Mathematics," Pylon 2 (2022). DOI: https://doi.org/10.48631/pylon.2022.2.92969.

Propylaeum
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§1 According to our reconstruction of the order of the surviving leaves of the fourth-century codex P.Math., the mathematical problem we designate as a3 (occupying leaf A verso lines 1–7, where the line number 7 is assigned to a diagram following the text) is the second of a series of twenty problems written in no apparent order between a model contract (a1) on leaf A recto and a metrological text (e3) on E verso. We repeat here the text and translation of this problem as they appear in our edition of the codex; the restorations of the lost beginnings and endings of lines are determined by the internal mathematical logic.[1]

A verso

– – – – –
[.....] πηχῶν [ϛ̅], πλάτος τακ̣[τύλων η̅,]
[πάχος δ]α̣κτήλων δ̅. οὕτω ποιοῦμ̣[εν. συντίθω]
[η̅ καὶ ϛ̅.] γί(νεται) ι̅δ̅. ὧν ἥμισυ ζ̅. ἐπὶ τὸ πλ̣[άτος, ζ̅ ἐπὶ]
[η̅.] γ̣ί(νεται) ν̅ϛ̅. ἐπὶ δὸ πάχος, τακτήλ̣[ων δ̅. γί(νεται) σ̅κ̅δ̅.]
[παρὰ] τ̣ὸν σ̅π̅η̅, καὶ τὰ ληπὰ εἰς [δακτύλους.]
[γί(νεται)] τ̣άκτη[λ]ο̣ι ι̅η̅ ω̅. οὕτως ἔχει ὁμ[οίως.]

1. *l.* δακτύλων | 2. *l.* δακτύλων | 4 *l.* τὸ | *l.* δακτύλων | 5 ληπὰ: π corr. ex γί(νεται), *l.* λοιπὰ | 6 *l.* δάκτυλοι

– – – – –
[. . .] 6 cubits, breadth [8] fingers,
[thickness] 4 fingers. We proceed as follows. [I add]
[8 and 6.] The result is 14. Half of this is 7. Times the breadth, [7 times]
[8.] The result is 56. Times the thickness, [4] fingers. [The result is 224.]
[(I divide) by] 288, and the remainder (converted) into [fingers].
[The result is] 18⅔ fingers. This way for similar cases.

§2 A typical problem text in P.Math. begins with a succinct statement of what we call the “scenario” of the problem, naming the object under consideration (say a plot of land or a cistern or a canal trench) and listing various given dimensions or other quantities pertaining to it. Then may come an explicit specification of the unknown quantity that one is required to calculate (say the plot’s area or the cistern’s capacity or the quantity of earth to be removed to make the trench), but this is often omitted as, presumably, obvious. The rest of the text, constituting the solution, will set out arithmetical operations on the given quantities that lead to the desired result. A diagram ostensibly representing the object of the scenario normally comes immediately below the text.

§3 In the present case, one or possibly two lines are broken away at the top of the leaf, so that we are missing part of the scenario. When the extant text begins, we are in the midst of the listing of the object’s given dimensions; the unknown quantity is left unspecified. The solution consists of two parts: (a) a sequence of arithmetical operations on the givens (lines 2–4) leading to the number 224—which is not provided with a name in relation to the object or with the units in which it is measured—followed by (b) further operations (lines 5–6) leading to the final result, 18⅔ fingers. Parallels in other problems in the codex (b5, c1, and g4), other papyri (P.Harris 50 and P.Oxy. 49

1 Bagnall and Jones 2019: 64–65 and 122–123; a pair of accidentally omitted brackets in the translation of line 2 are restored here.

3455), and a metrological treatise ascribed to Didymus of Alexandria show that these last operations are a metrological conversion of a volume measured in quasi-units of one linear cubit by one linear finger by one linear finger (called χυδε̂οι or χοιδε̂οι—scil. χυδαι̂οι—δάκτυλοι in other sources but not P.Math.) into volume units that remain unnamed but that are subdivided, on analogy with linear cubits, into 24 volumetric fingers.[2] The constant factor 288 used for this conversion indicates that the volume unit in question is equivalent to half a cubic cubit, or as Lougovaya names it, a "three-palm solid cubit," in distinction from a "two-palm solid cubit" for which the conversion constant was 192.[3] Lougovaya convincingly argues that both these volume units were used specifically for measuring quantities of wood.

§4 When part or all of the scenario of a problem text is missing, we can usually achieve at least a partial reconstruction on the basis of information contained in the solution and diagram. In particular, this genre of mathematics was algorithmic in nature: the student was expected to learn a repertoire of algorithms for different kinds of scenario, givens, and unknown, and to identify and apply the appropriate algorithm using the given quantities. In our edition of P.Math. we assembled a reference list—not claiming to be exhaustive—of nearly forty algorithms attested in Greek mathematical papyri.[4] The algorithm employed in the solution of a problem ought naturally to be one that is suitable for the scenario.

§5 We believed we could recognize in part (a) of the solution of problem a3 a pair of algorithms that, taken together, are appropriate for finding the volume of a solid prismoidal shape having a trapezoidal cross-section where we have been given the "lengths" of the two unequal parallel sides of the cross-section, the "breadth" measured across the cross-section perpendicular to the parallel sides, and the "thickness" perpendicular to the cross-section.[5] The first is our algorithm P3A for finding the area of a planar trapezoidal figure, given the two lengths and the breadth: take the average of the two lengths and multiply by the breadth. The second, our algorithm S2, is a general algorithm for finding the volume of any prism-like solid given the area of its unvarying cross-section and its thickness: multiply the cross-section's area by the thickness. Working backwards from the calculations, it seemed straightforward that the 6 cubits in line 1 should represent the shorter of the two lengths, and that the longer length had to have been 8 cubits, since their average is 7 cubits as stated in the text. Hence the object, whatever it was, appeared to be shaped like long board with beveled ends, as shown in the reconstructed diagram Fig. 1 reproduced from our commentary:

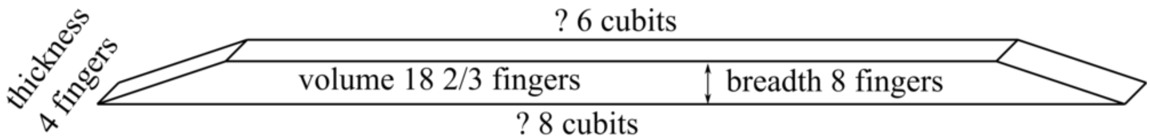

Fig. 1. The object of a3 reconstructed as a rectangular prism with beveled ends.

§6 The other problems in P.Math. that employ the two-palm and three-palm solid cubits all concern rectilinear, stick-like shapes, without beveled ends. In b5, the cross-section is a trapezoid; in c1, it is an irregular quadrilateral. The scenario of g4 confusingly specifies a circular cross-section but gives dimensions appropriate for a rectangular cross-section; here, the solution treats the cross-section as a rectangle, but the diagram shows it as a circle. The diagrams in P.Math. frequently exhibit a somewhat wayward relation to the geometrical scenarios of the problems that they accompany, and so

2 Lougovaya 2022.

3 1 palm is equivalent to 1/6 cubit, so a rectangular solid with dimensions 1 cubit × 1 cubit × 3 palms is half a cubic cubit. (See Lougovaya 2022, Fig. 2 at §11.) Since 1 cubic cubit contains 24×24 (i.e. 576) of the 1 cubit × 1 finger × 1 finger quasi-units, the number of such quasi-units in a half cubic cubit is 288.

4 Bagnall and Jones 2019: 27–49.

5 Bagnall and Jones 2019: 122–123. The algorithms in question are our P3A (trapezoid area), p. 30, and S2 (prism volume), pp. 41–42.

we presumed that the damaged diagram that follows a3 (redrawn in Fig. 2), which appears to have shown two crudely executed concentric circles, was “an erroneous intrusion.”

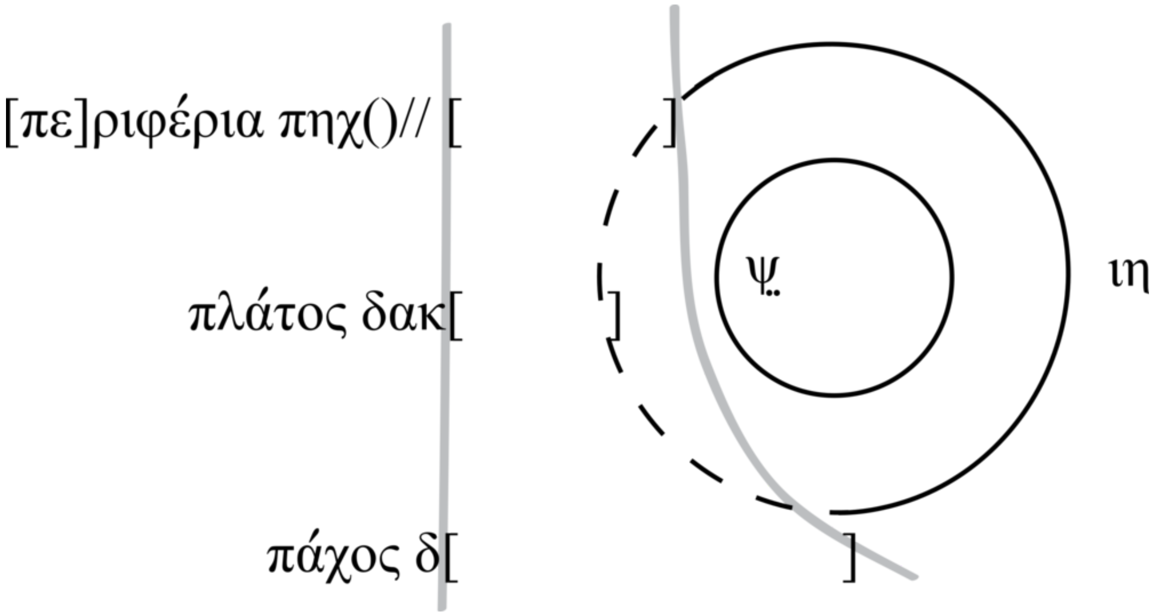


Fig. 2. Redrawing of the diagram following problem a3 on leaf A verso. Gray lines approximate broken edges of the papyrus, and broken lines are our restorations.

§7 In a session (May 31, 2022) of the seminar “Lectures de textes mathématiques anciens” of the Laboratoire SPHERE, Université Paris Cité, during which one of us (Jones) presented selections from P.Math., Karine Chemla pointed out that in the ancient Chinese mathematical text *Nine Chapters* the problem of finding the area of a two-dimensional figure bounded externally and internally by two concentric circles, given the outer and inner circumferences and the “transverse diameter” (i.e. the difference between the circles’ radii) is treated analogously to the algorithm for a trapezoidal area. Further discussion among the seminar’s participants (to whom full credit for the outcome is due) established that if a3 was understood as a problem of finding the volume of a ring-shaped solid bounded by coaxial cylindrical surfaces and by parallel planes at right angles to the cylinders’ common axis, the text and diagram would make almost perfect sense together as they stand in the manuscript.[6]

§8 The two relevant problems in the *Nine Chapters* are those numbered 1.37 and 1.38 in the edition of Chemla and Guo, quoted below without the interspersed commentaries.[7] Note that in these texts *BU* is both a unit of length and a unit of area (equivalent to a square with side one linear *BU*) while *MU* is an area unit equivalent to 240 (area) *BU*.

§9 (1.37) Let us assume that one has a field in the shape of a ring whose interior circumference is 92 *BU*, exterior circumference 122 *BU*, and transverse diameter 5 *BU*. It is asked how much is the field. Answer: 2 *MU* 55 *BU*.

§10 (1.38) Let us again assume that one has a field in the shape of a ring whose interior circumference is 62 3/4 *BU*, exterior circumference 113 1/2 *BU*, and transverse diameter 12 2/3 *BU*. It is asked how much is the field. Answer: 4 *MU* 156 1/4 *BU*.

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6 The diagram, so far as it survives, seems to have named just one “circumference,” and the numeral 18 is presumably the solution lacking its fraction.

7 Chemla and Guo 2004: 192–195 with 782–783. Our translation is of their French translation.

§11 Procedure: One adds interior and exterior circumferences, and one takes the half of this. One multiplies this by the transverse diameter, which makes the *BU* of the product.

§12 These problems highlight the fact that the "transverse diameter" is not actually independent of the two given circumferences, though it is treated in the statement of the problem as a separate given. Supposing the common inexact algorithm for finding the diameter of a circle of given circumference by dividing the circumference by 3,[8] the transverse diameter should be one-sixth of the difference between the exterior and interior circumferences, as indeed it is in 1.37 but not, as the 3rd century CE commentator Liu Hui observed, in 1.38; hence Liu Hui interpreted 1.38 as concerning an *incomplete* ring-shaped field, namely the segment of the complete ring bounded by two radii (see Figs. 3 and 4). (For this interpretation the "circumferences" would mean the lengths of the two arcs bounded by the radii, not the complete circles.) In any event, if one applies the general "procedure" stated after 1.38 to the givens in both problems, one obtains exactly the stated solutions.

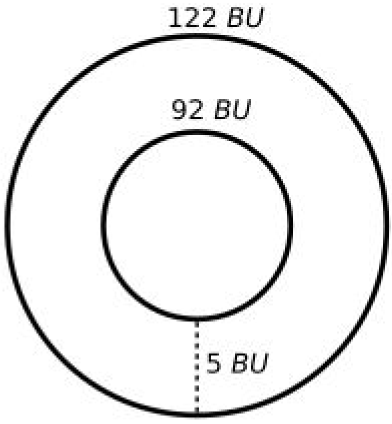


Fig. 3. Reconstructed diagram for *Nine Chapters* 1.37 (not to scale).

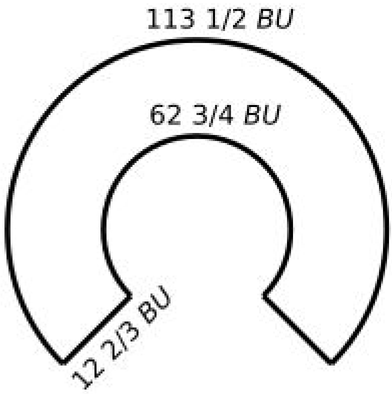


Fig. 4. Reconstructed diagram for *Nine Chapters* 1.38 according to Liu Hui's interpretation (not to scale).

§13 Our problem a3 turns out to be closely analogous to *Nine Chapters* 1.37, with the difference that the object in a3 is a three-dimensional one having "thickness" in addition to the dimensions that determine its ring-shaped faces (Fig. 5). Thus the "breadth" in a3 is the counterpart of the "transverse diameter" in the Chinese problem, and here too it is presented as one of the givens although its value is a consequence of the choice of circumferences, 8 fingers being (as expected) one-sixth of the difference

8 Our algorithm P9i (circle inverse circumference algorithm), Bagnall and Jones 2019: 38.

between 8 and 6 cubits. As usual with such problems about slab-like or prismoidal solid shapes, the area of the face is first calculated and then multiplied by the thickness.

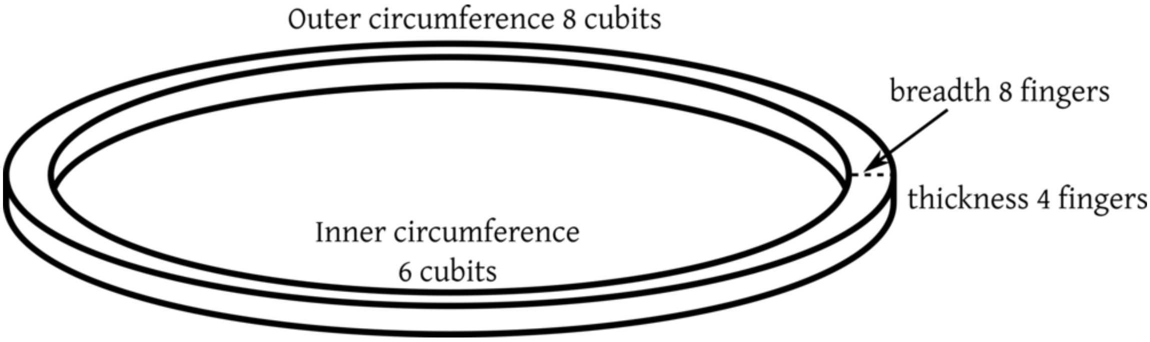


Fig. 5. The solid ring of problem a3.

§14 So far as we are aware, this is the first instance found in a Greek mathematical papyrus (taken in the broad sense, including other media such as wooden boards) of a problem concerning a three-dimensional ring-shaped figure. We have not found problems concerning the complete planar ring in papyri either, though Heron, *Metrica* 1.26 deduces a different algorithm for finding the area given the diameters of the bounding circles.[9] In the first problem in MPER NS 15 178 (= P. Vind. inv. G 26740), however, we find an analogue to Liu Hui's interpretation of *Nine Chapters* problem 1.38 as a determination of the area of a segment of a ring-shaped area, here called a μηνίσκος, with the same algorithm.[10] (The givens are called ἐκτὸς/ἐντὸς περίμετρος and βάσις, paralleling the terminology of the Chinese text.) More remotely, one can point to kinship between the planar and solid ring algorithms and algorithms for finding the surface area of a conical frustum (without its planar faces) or the number of seats in a theater that is hypothesized as a series of discrete horizontal rows of discrete seats distributed on a conical surface (e.g. the problem on the front side of PSI 3 186), since a planar ring can be thought of as the curved surface of a conical frustum.[11]

§15 We thus supplement our collection of algorithms with the following:

§16 **P11A. Planar ring area algorithm (approximate).** Given a planar ring bounded by two concentric circles with circumferences $c_1$ and $c_2$ and the difference between their radii $d$, to find the area $A$:

(i) Add $c_1$ and $c_2$.
(ii) Divide the sum by 2.
(iii) $A$ is the quotient multiplied by $d$

$$A = \frac{(c_1 + c_2)}{2} \times d$$

Used in: P.Math. a3 (together with S2).

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9 Acerbi and Vitrac 2014: 214–215, and see also *Metrica* 3.9 and Geometrica 21.

10 Bruins, Sijpesteijn, and Worp 1974: 298–299.

11 We thank Julia Lougovaya for pointing this out for PSI 3 186.

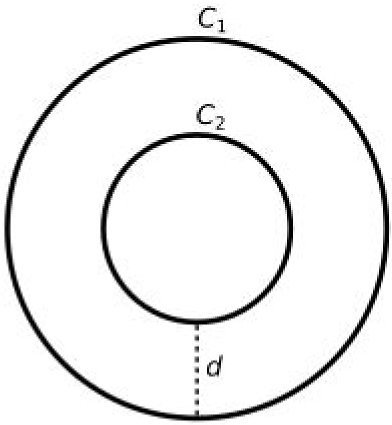


Fig. 6. Diagram for algorithm P11A.

§17 **P11B. Planar ring segment area algorithm.** Given a segment of a planar ring bounded by two concentric circular arcs with lengths $c_1$ and $c_2$ and two radial lines of length $d$, to find the area $A$:

(i) Add $c_1$ and $c_2$.
(ii) Divide the sum by 2.
(iii) $A$ is the quotient multiplied by $d$

$$A = \frac{(c_1 + c_2)}{2} \times d$$

Used in: MPER NS 15 178.1.1–10.

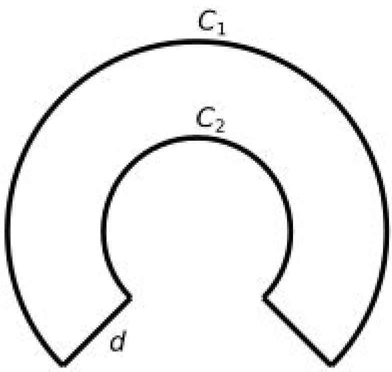


Fig. 7. Diagram for algorithm P11B.

§18 Chemla and Guo hypothesize that the algorithm for the ring-shaped region (complete or segment) could have been derived by imagining a deformation of the ring, cut along a radius, into a trapezoid by straightening the circumferences into parallel lines (fig. 2). Interestingly, however, the algorithm is geometrically exact when applied to a segment, and only inexact for the complete ring insofar as the assumption that the difference of the circles' radii is one-sixth the difference of the circumferences is inexact, implying $\pi = 3$. Hence the algorithm could also be deduced from the direct approach of calculating the areas of the circles separately and subtracting the smaller from the larger, on analogy with what Heron does in *Metrica* 1.26. To the attestations of the algorithm in texts from China and Greco-Roman Egypt, one can add a late Babylonian tablet W 23291-x in which are calculated the areas of a series of concentric planar rings whose circumferences form an arithmetical sequence, and likely also an Old Babylonian school tablet CUNES 52-02-043 that represents a similar problem in diagrammatic form without stating the steps of the solutions.[12]

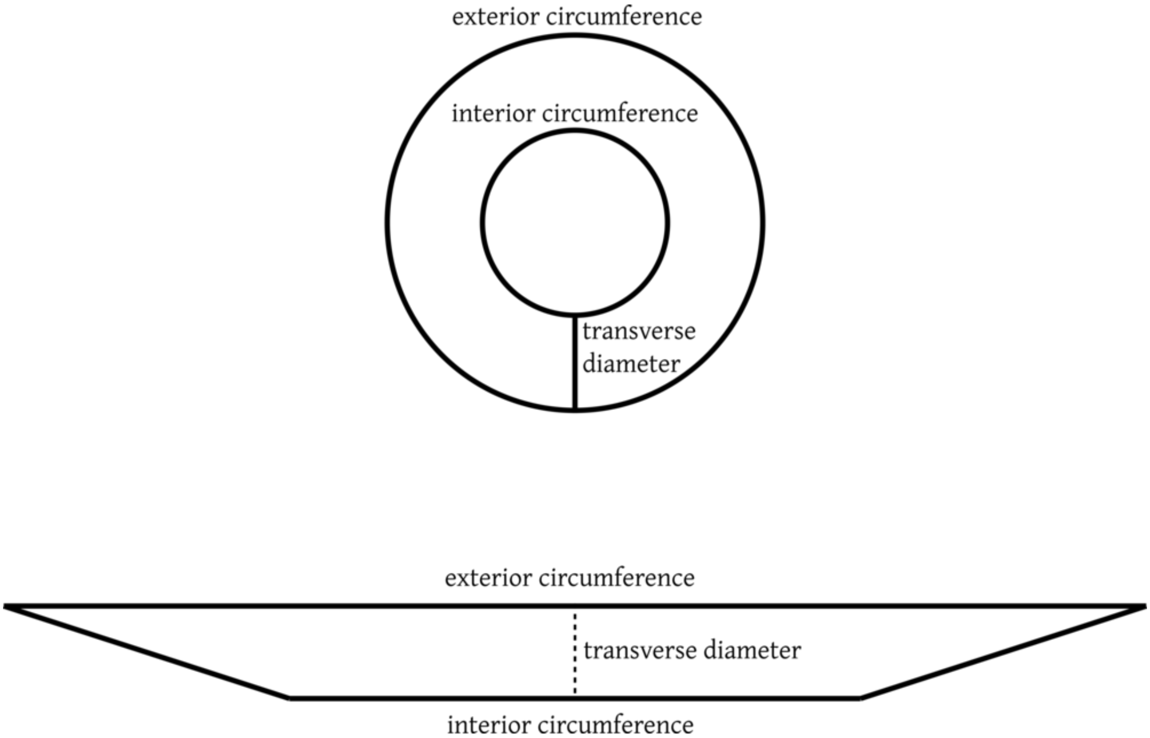


Fig. 8. Transformation of a planar ring into a trapezoid.

§19 What was the object named in the lost beginning of a3, having the form of a flat ring something over a meter broad and a rectangular cross-section twice as broad as it was thick? A capping of a circular well or cistern? Or better, since the metrology appears to have been particularly associated with wooden objects, the felloe or outer rim (ἴτυς) of a large wheel? In Heron, *Metrica* 1.26 the planar ring is called ἴτυς, while in his mechanical writings Heron uses the same word for annular components in a water-heating device (*Pneumatica* 3.34) and an automatic theater (*De Automatis* 16) where it serves as a revolving platform for mythological figurines. Our best guess is that this might have been the word that introduced our problem, so we tentatively restore the problem and its diagram thus:

[ἴτυς. ἐκτὸς περιφέρεια πηχῶν η̅, ἐντὸς περι]-
[φέρεια] πηχῶν [ϛ̅], πλάτος τακ̣[τύλων η̅,]
[πάχος [δ]ακτήλων δ̅. οὕτω ποιοῦμ̣[εν. συντίθω]
[η̅ καὶ ϛ̅.] γί(νεται) ι̅δ̅. ὧν ἥμισυ ζ̅. ἐπὶ τὸ πλ[άτος, ζ̅ ἐπὶ]
[η̅.] γί(νεται) ν̅ϛ̅. ἐπὶ δὸ πάχος, τακτήλ̣[ων δ̅. γί(νεται) σ̅κ̅δ̅.]
[παρ̣ὰ] τὸν σ̅π̅η̅, καὶ τὰ ληπὰ εἰς [δακτύλους.]
[γί(νεται)] τ̣άκτη[λ]ο̣ι ι̅η̅ ϖ̅. οὕτως ἔχει ὁμ[οίως.]

1 *l.* δακ[τύλων] 2 *l.* [δ]ακτύλων 4 *l.* τὸ *l.* δακτύλ[ων] 5 *l.* λοιπὰ 6 *l.* δάκτυ[λ]οι

12 Friberg, Hunger, and al-Rawi 1990; Brunke 2018. See also Friberg 2005: 197–198.

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Bagnall, Roger S.
GND: https://d-nb.info/gnd/124747213
ORCID: https://orcid.org/0000-0002-9817-3326
ISAW, New York
rsb331@nyu.edu

Jones, Alexander
GND: https://d-nb.info/gnd/136112129
ORCID: https://orcid.org/0000-0002-1538-8410
ISAW, New York
alexander.jones@nyu.edu